Hermitian conjugation

Given operator $\hat{A}$, there exists another operator $\hat{A}^{+}$ (called hermitian conjugate of $\hat{A}$), such that

if $\hat{A}|\psi\rangle = |\chi\rangle$ then $\langle\chi| = \langle\psi|\hat{A}^{+}$

Corollary: $\langle\psi|\hat{A}^{+}|\phi\rangle = \langle\chi|\phi\rangle = \langle\phi|\chi\rangle^{*} = \langle\phi|\hat{A}|\psi\rangle^{*}$ (*)

Claim: $(\hat{A}^{+})^{+} = \hat{A}$ [prove]

If $\hat{A} = c\hat{\mathbb{1}}$ then $\hat{A}^{+} = c^{*}\mathbb{1}$ [prove] (hermitian conj analog of complex conj)

$(\hat{B}\hat{A})^{+} = \hat{A}^{+}\hat{B}^{+}$ ([illegible]) (**)

Proof: let $|\omega\rangle = \hat{B}\hat{A}|\psi\rangle$ then $\langle\omega| = \langle\psi|(\hat{B}\hat{A})^{+}$

Now $|\omega\rangle = \hat{B}(\hat{A}|\psi\rangle) = \hat{B}|\chi\rangle \Rightarrow \langle\omega| = \langle\chi|\hat{B}^{+}$

But $|\chi\rangle = \hat{A}|\psi\rangle \Rightarrow \langle\chi| = \langle\psi|\hat{A}^{+}$ so

$\langle\omega| = (\langle\psi|\hat{A}^{+})\hat{B}^{+} = \langle\psi|\hat{A}^{+}\hat{B}^{+}$ QED

$$\langle\phi|\hat{A}\hat{B}|\psi\rangle^{*} = \langle\psi|\hat{B}^{+}\hat{A}^{+}|\phi\rangle$$ ← use (*) and (**)

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hermitian conjugation ⇒ ket ⟺ bra
$A \to A^{+}$
reverse order of objects

$\boxed{\text{Hermitian operators}}$

An operator $\hat{A}$ is Hermitian if $\hat{A}^\dagger = \hat{A}$

(anti Hermitian if $\hat{A}^\dagger = -\hat{A}$)

All observables correspond to Hermitian operators

[E.g. $\hat{S}_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$, $\hat{S}_z$: ket $\leftrightarrow$ bra, so same]

E.g. energy E corresponds to Hamiltonian $\hat{H}$

Assume (for now) ~~For a bound state~~, $\hat{H}$ has discrete eigenvalues E_n (results of measurement)

$\hat{H}|E_n\rangle = E_n|E_n\rangle$ $\quad$ $|E_n\rangle$ = states of determinate energy E_n

Assume $|E_n\rangle$ are normalized: $\langle E_n|E_n\rangle = 1$

Two key results: a Hermitian operator has

① real eigenvalues (observables are real)

② orthogonal eigenstates

We'll prove these for $\hat{H}$, but results are general

① $E_n = \langle E_n|\hat{H}|E_n\rangle$

$E_n^* = \langle E_n|\hat{H}|E_n\rangle^* = \langle E_n|\hat{H}^\dagger|E_n\rangle = \langle E_n|\hat{H}|E_n\rangle = E_n$

(2) Eigenstates corresponding to __distinct__ eigenvalues are orthogonal

Consider $\langle E_m | \hat{H} | E_n \rangle = E_n \langle E_m | E_n \rangle$

Take c.c. $\langle E_n | \hat{H}^{+} | E_m \rangle = E_n^{*} \langle E_n | E_m \rangle$

$$\langle E_n | \hat{H} | E_m \rangle = E_n \langle E_n | E_m \rangle$$

$$E_m \langle E_n | E_m \rangle = E_n \langle E_n | E_m \rangle$$

$$(E_m - E_n) \langle E_n | E_m \rangle = 0$$

Since $E_m \neq E_n \Rightarrow \langle E_n | E_m \rangle = 0$

If eigenstates are degenerate, proof fails.
But an orthogonal basis can always be chosen
by the Gram-Schmidt orthonormalization procedure

e.g. assume $\langle E_n^1 | E_n^2 \rangle \neq 0$ where $\begin{cases} \hat{H} | E_n^1 \rangle = E_n | E_n^1 \rangle \\ \hat{H} | E_n^2 \rangle = E_n | E_n^2 \rangle \end{cases}$

Define $| \tilde{E}_n^2 \rangle = | E_n^2 \rangle - \langle E_n^1 | E_n^2 \rangle | E_n^1 \rangle$

$\hat{H} | \tilde{E}_n^2 \rangle = E_n | \tilde{E}_n^2 \rangle$ still an eigenstate ✓

$\langle E_n^1 | \tilde{E}_n^2 \rangle = 0$

Normalize $| \tilde{E}_n^2 \rangle \Rightarrow | \hat{\tilde{E}}_n^2 \rangle$

Then $| E_n^1 \rangle$ and $| \hat{\tilde{E}}_n^2 \rangle$ form an orthonormal eigenbasis

[HW problem]

We can safely assert that there exists a basis s.t.

$$\langle E_n | E_m \rangle = \delta_{nm} \qquad \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

We assume that eigenstates of $\hat{H}$ are complete, i.e. any state $|\psi\rangle$ can be written as a linear combination

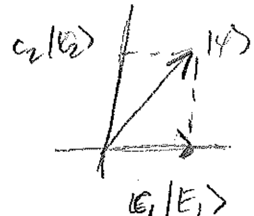
$$|\psi\rangle = \sum c_n |E_n\rangle$$

Since basis is orthonormal, $c_n = \langle E_n|\psi\rangle$.

check: $\langle E_n|\psi\rangle = \sum_n c_m \langle E_n|E_m\rangle = \sum_n c_m \delta_{nm} = c_n$

Define projection operator $\hat{\Pi}_n$ as $\Pi_n|\psi\rangle = c_n|E_n\rangle$

Measurement of E_n collapses $|\psi\rangle$ to projection onto $|E_n\rangle$.



Then $\hat{\Pi}_n = |E_n\rangle\langle E_n|$

Easy to verify: $\hat{\Pi}_n^2 = \hat{\Pi}_n$

$\hat{\Pi}_n\hat{\Pi}_m = 0 \quad n \neq m$

Completeness: $\sum_n \hat{\Pi}_n = \boxed{\sum_n |E_n\rangle\langle E_n| = \mathbb{1}}$ ← emphasize. if basis is not complete, can't build $|\psi\rangle$ back up out of $|E_n\rangle$

Provided $|\psi\rangle$ is normalized the Prob. of obtaining E_n is given by

$$\langle\psi|\hat{\Pi}_n|\psi\rangle = \underbrace{\langle\psi|E_n\rangle}_{c_n^*}\underbrace{\langle E_n|\psi\rangle}_{c_n} = |c_n|^2$$

Check that probabilities add up to 100%:

$$\sum_n |c_n|^2 = \sum_n \langle\psi|\hat{\Pi}_n|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \langle\psi|\psi\rangle = 1$$

(HW) Show that $\hat{H} = \sum E_n \hat{\Pi}_n$

(Two operators are equal if they give the same result when acting on an arbitrary ket $|\psi\rangle$.)

(HW) show $\langle\psi|\hat{H}|\psi\rangle = \langle E\rangle$

(HW) show $\langle\psi|\hat{H}^2|\psi\rangle = \langle E^2\rangle$

(HW) show If $|\chi\rangle = (\hat{H} - \langle E\rangle)|\psi\rangle$, then $\langle\chi|\chi\rangle = (\Delta E)^2$

[Hint: one can insert $\hat{1} = \sum_n \hat{\Pi}_n$ at an arbitrary point

eg $\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\hat{1}\,\hat{H}\,\hat{1}|\psi\rangle$]

An eigenstate of $\hat{H}$ has zero uncertainty. $\Delta E = 0$

Conversely, a state of zero uncertainty is an eigenstate

Proof: $\Delta E = 0 \Rightarrow \langle\chi|\chi\rangle = 0 \Rightarrow |\chi\rangle = 0$

$\Rightarrow \hat{H}|\psi\rangle = \langle E\rangle|\psi\rangle \Rightarrow$ eigenstate