Reintroduce spin

electron spin $\vec{S}$ commutes w/ H, $\vec{L}$

Complete set of commuting operators: H, L^2, S^2, L_3, S_3

Goal: find simultaneous eigenstates $|n, l, m_l, m_s\rangle$

$$\begin{cases} H|n,l,m_l,m_s\rangle = E_n|n,l,m_l,m_s\rangle \\ L^2|n,l,m_l,m_s\rangle = \hbar^2 l(l+1)|n,l,m_l,m_s\rangle \\ S^2|n,l,m_l,m_s\rangle = \frac{3}{4}\hbar^2|n,l,m_l,m_s\rangle \\ L_3|n,l,m_l,m_s\rangle = \hbar m_l|n,l,m_l,m_s\rangle \\ S_3|n,l,m_l,m_s\rangle = \hbar m_s|n,l,m_l,m_s\rangle \end{cases}$$

Define $\chi_{m_s} = \begin{cases} \begin{pmatrix}1\\0\end{pmatrix} & \text{for } m_s = \frac{1}{2} \\ \begin{pmatrix}0\\1\end{pmatrix} & \text{for } m_s = -\frac{1}{2} \end{cases}$

In position space, eigenfunctions are

$$u_{n l m_l m_s}(r,\theta,\phi) = R_{nl}(r)\, Y_{l m_l}(\theta,\phi)\, \chi_{m_s}$$

$$= (\text{const})\, r^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na_0}\right) e^{-\frac{r}{na_0}}\, Y_{l m_l}(\theta,\phi)\, \chi_{m_s}$$

$n = 1, 2, 3 \ldots$

$l = 0, \ldots, n-1$

$m_l = -l, \ldots, l$

$m_s = -\frac{1}{2}, \frac{1}{2}$

$$E_n = -\frac{1}{2}\frac{mc^2\alpha^2}{n^2}$$

$n=3$	$\frac{3S}{2}$	$\frac{3P}{6}$	$\frac{3D}{10}$
$n=2$	$\frac{2S}{2}$	$\frac{2P}{6}$	
$n=1$	$\frac{1S}{2}$		
$l=$	0	1	2

But recall total angular momentum $\vec{J} = \vec{L} + \vec{S}$

J^2 commutes w/ L^2 and S^2 but not w/ L_3 and S_3 (unless $l=0$)

$\Rightarrow$ $u_{n l m_l m_s}$ are not eigenstates of J^2.

Instead, choose H, L^2, S^2, J^2, J_3 as complete set of comm ops.

w/ eigenstates $\psi_{n l j m_j}$ where $J^2 \psi_{nljm_j} = \hbar^2 j(j+1) \psi_{nlj}$

$J_3 \psi_{nljm_j} = \hbar m_j \psi_{nljm_j}$

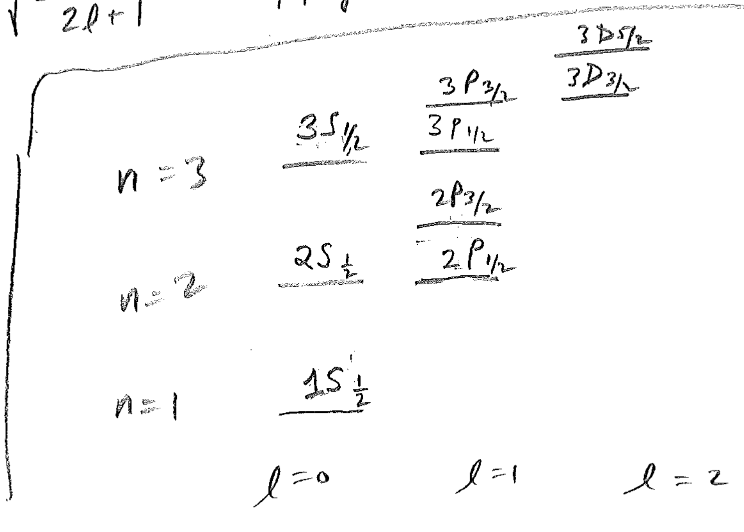
Recall (spin l) $\otimes$ (spin $\frac{1}{2}$) = spin $(l+\frac{1}{2})$ $\oplus$ spin $(l-\frac{1}{2})$

so $j = l \pm \frac{1}{2}$

Use Clebsch-Gordan coeffs to find [Sakurai, p.309; Gasiorowicz]

$$\psi_{n,l,j=l+\frac{1}{2},m_j} = \sqrt{\frac{l+\frac{1}{2}+m_j}{2l+1}}\, u_{n,l,m_j-\frac{1}{2},\frac{1}{2}} + \sqrt{\frac{l+\frac{1}{2}-m_j}{2l+1}}\, u_{n,l,m_j+\frac{1}{2},-\frac{1}{2}}$$

$$\psi_{n,l,l-\frac{1}{2},m_j} = -\sqrt{\frac{l+\frac{1}{2}-m_j}{2l+1}}\, u_{n,l,m_j-\frac{1}{2},\frac{1}{2}} + \sqrt{\frac{l+\frac{1}{2}+m_j}{2l+1}}\, u_{n,l,m_j+\frac{1}{2},-\frac{1}{2}}$$



Addition of orbital and spin angular momentum

$$J^2 = L^2 + S^2 + 2L_zS_z + L_+S_- + L_-S_+$$

$$L_\pm Y_{lm} = \sqrt{(l \pm m + 1)(l \mp m)}\, \hbar\, Y_{l,m\pm 1}$$

$$S_\pm \chi_\mp = \hbar\, \chi_\pm$$

$$\psi_{j,m+\frac{1}{2}} = \alpha Y_{lm}\chi_+ + \beta Y_{l,m+1}\chi_-$$

$$\begin{aligned} J^2 \psi_{j,m+\frac{1}{2}} &= \alpha\left[l(l+1) + \tfrac{3}{4} + 2m\left(\tfrac{1}{2}\right)\right]\hbar^2 Y_{lm}\chi_+ \\ &+ \alpha\left[\sqrt{(l+m+1)(l-m)}\right]\hbar^2 Y_{l,m+1}\chi_- \\ &+ \beta\left[l(l+1) + \tfrac{3}{4} + 2(m+1)\left(-\tfrac{1}{2}\right)\right]\hbar^2 Y_{l,m+1}\chi_- \\ &+ \beta\sqrt{(l-m)(l+m+1)}\,\hbar^2 Y_{l,m}\chi_+ \\ &= j(j+1)\hbar^2\left[\alpha Y_{lm}\chi_+ + \beta Y_{l,m+1}\chi_-\right] \end{aligned}$$

$$\Rightarrow \alpha\left(l^2 + l + \tfrac{3}{4} + m\right) + \beta\sqrt{(l-m)(l+m+1)} = \alpha j(j+1)$$

$$\alpha\sqrt{(l+m+1)(l-m)} + \beta\left(l^2 + l - \tfrac{1}{4} - m\right) = \beta j(j+1)$$

$$j = l + \tfrac{1}{2} \Rightarrow \quad \beta\sqrt{(l-m)(l+m+1)} = \alpha(l-m)$$

$$\psi_{l+\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{l+m+1}{2l+1}}\, Y_{lm}\chi_+ + \sqrt{\frac{l-m}{2l+1}}\, Y_{l,m+1}\chi_-$$

$$j = l - \tfrac{1}{2} \Rightarrow \quad \beta\sqrt{(l-m)(l+m+1)} = -(m+l+1)\alpha$$

$$\psi_{l-\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{l-m}{2l+1}}\, Y_{lm}\chi_+ - \sqrt{\frac{m+l+1}{2l+1}}\, Y_{l,m+1}\chi_-$$

$n=2$

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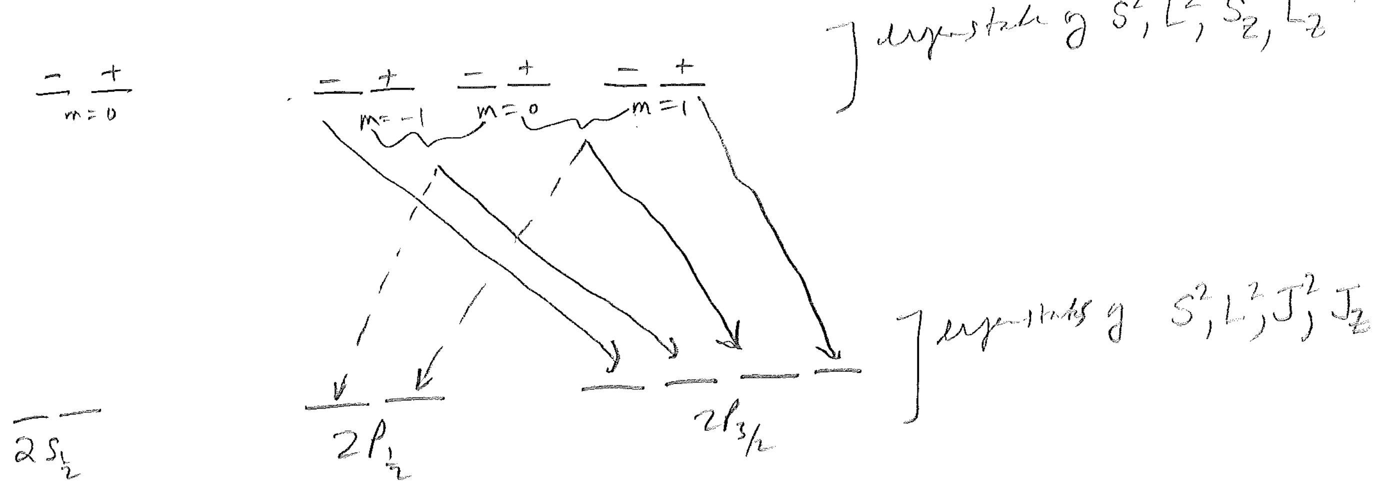

Recall from prob

$$\overset{j\ \ m_j}{|\tfrac{3}{2},\tfrac{3}{2}\rangle} = \overset{m_\ell\ \ m_s}{|1;\tfrac{1}{2}\rangle} = Y_{11}\,\chi_{\frac{1}{2}}$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0;\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1;-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,Y_{10}\,\chi_{\frac{1}{2}} + \sqrt{\tfrac{1}{3}}\,Y_{11}\,\chi_{-\frac{1}{2}}$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0;-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|-1;\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |-1;-\tfrac{1}{2}\rangle.$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|0;\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|0;-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|-1;-\tfrac{1}{2}\rangle$$