55-1

~~Hydrogen atom~~ Particle in central potential

Reduction of 2-body problem to 1-body problem

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

$$V = V(\vec{r}_1 - \vec{r}_2)$$

skipped in 2013

Let $\vec{r}_{cm} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$

$$M = m_1 + m_2$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} \qquad m = \text{reduced mass}$$

Then $T = \frac{1}{2} M \dot{\vec{r}}_{cm}^{\,2} + \frac{1}{2} m \dot{\vec{r}}^{\,2}$

$$V = V(\vec{r})$$

$$H = \underbrace{\frac{\vec{p}_{cm}^{\,2}}{2M}}_{\text{"free particle" (hydrogen cm)}} + \underbrace{\frac{\vec{p}^{\,2}}{2m} + V(\vec{r})}_{\text{particle in a 3d potential}}$$

If $m_1 \gg m_2$, then $m \approx m_2$ eg electron

$|E, l, m\rangle$ = simultaneous eigenstate of H, L^2, L_3

$$u_{Elm}(r,\theta,\phi) = R_{Elm}(r)\, Y_{lm}(\theta,\phi)$$

$$H\, u_{Elm} = E\, u_{Elm}$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{\text{You will show this } = -\frac{L^2}{\hbar^2}}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\right]R_{Elm}Y_{lm} + V(r)R_{Elm}Y_{lm} = E R_{Elm}Y_{lm}$$

$$-\frac{\hbar^2}{2m}\left[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{Elm}}{\partial r}\right)}_{=\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR_{Elm})} - \frac{l(l+1)}{r^2}R_{Elm}\right] + V(r)R_{Elm} = E R_{Elm}$$

$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rR_{Elm})$, easy to verify

mass, not magnetic qu no

$$\frac{d^2}{dr^2}(rR_{Elm}) - \frac{l(l+1)}{r^2}(rR_{Elm}) = -\frac{2m}{\hbar^2}(E - V(r))(rR_{Elm})$$

radial Schrödinger eqn

N.B. Coefficients of the eqn depend on l, but not m. therefore eigenvalues E + eigenfunctions $R(r)$ do not depend on m i.e. the $(2l+1)$ eigenfunctions

$$u_{Elm}(r,\theta,\phi) = R(r)\, Y_{lm}(\theta,\phi) \qquad m=-l,\dots,l$$

are degenerate $\Rightarrow$ this is due to spherical symmetry

[Drop m from label for $R_{El}(r)$]

$$u_{Elm}(r,\theta,\phi) = R_{El}(r)\, Y_{lm}(\theta,\phi)$$

E only depends on l