Define position space eigenfunctions of H, L^2, L_3:

$$u_{E\ell m}(r,\theta,\phi) \equiv \langle \vec{x} | E, \ell, m \rangle$$

We will find that eigenfunction is separable

$$u_{E\ell m}(r,\theta,\phi) = R_{E\ell m}(r)\, Y_{\ell m}(\theta,\phi)$$

$L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$ so L_3 eigenvalue eqn is:

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\left[R_{E\ell m}(r)\, Y_{\ell m}(\theta,\phi)\right] = \hbar m\, R_{E\ell m}(r)\, Y_{\ell m}(\theta,\phi)$$

$$\frac{\partial Y_{\ell m}}{\partial\phi} = im\, Y_{\ell m}$$

$$\frac{dY_{\ell m}}{Y_{\ell m}} = im\, d\phi$$

$$\ln Y_{\ell m} = im\phi + (\text{arb. fcn of } \theta)$$

$$Y_{\ell m} = (\text{func of } \theta)\, e^{im\phi}$$

spatial wavefunction must be single valued

$$Y_{\ell m}(\theta, \phi + 2\pi) = Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow \quad e^{im(\phi+2\pi)} = e^{im\phi}$$

$$e^{2\pi i m} = 1 \quad \Rightarrow \quad m \in \mathbb{Z} \text{ (not half int.)} \Rightarrow \ell \in \mathbb{Z}$$

m_{max} = ℓ

[Bohr!]

orbital ang. mom. ⇒ integer valued

In Hw

You calculated

$$L_1 = i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$L_2 = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

Define $L_\pm = L_1 \pm i L_2$

$$L_\pm = i\hbar\left(\underbrace{[\sin\phi \mp i\cos\phi]}_{\mp i(\underbrace{\cos\phi \pm i\sin\phi}_{e^{\pm i\phi}})}\frac{\partial}{\partial\theta} + \cot\theta\underbrace{[\cos\phi \pm i\sin\phi]}_{e^{\pm i\phi}}\frac{\partial}{\partial\phi}\right)$$

$$= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$L_+ \uparrow$
- Y_{ll}
- $Y_{l,l-1}$
- $\vdots$
- $Y_{l,-l}$

$$L_+ Y_{ll} = 0$$

$$Y_{ll}(\theta,\phi) = g(\theta)e^{il\phi}$$

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(g(\theta)e^{il\phi}\right) = 0$$

$$\frac{dg}{d\theta} + i\cot\theta\,(il)\,g(\theta) = 0$$

$$\frac{dg}{g} = l\cot\theta\,d\theta = l\frac{\cos\theta\,d\theta}{\sin\theta}$$

$$\ln g = l\ln\sin\theta + \text{const}$$

$$g(\theta) = C(\sin\theta)^l$$

$$\Rightarrow Y_{ll} = C\,(\sin\theta)^l e^{il\phi}$$

$Y_{\ell m}(\theta,\phi)$ are normalized by

$$\int |Y_{\ell m}(\theta,\phi)|^2 \underbrace{d\Omega}_{\text{solid angle}} = 1$$

$$\int |Y_{\ell m}(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi = 1$$

$$\int |Y_{\ell\ell}|^2 \sin\theta \, d\theta \, d\phi = |C|^2 \underbrace{\int_0^\pi \sin^{2\ell+1}\theta \, d\theta}_{\frac{2(2\ell)!!}{(2\ell+1)!!}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\Rightarrow C = \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!}\frac{1}{4\pi}} \cdot \underbrace{(-1)^\ell}_{\text{convention}}$$

$$Y_{00} = +\frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\phi}$$

$\vdots$

$$\int_0^\pi \sin^{2\ell+1}\theta \, d\theta = \int_{-1}^1 (1-x^2)^\ell \, dx = \begin{cases} \ell=0 : 2 \\ \ell=1 : \frac{4}{3} \\ \ell=2 : \frac{16}{15} \end{cases}$$

Recall $L_{\pm} Y_{\ell m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\, \hbar\, Y_{\ell, m\pm 1}$

$$Y_{\ell,\ell-1} = \frac{1}{\sqrt{2\ell}} \frac{L_-}{\hbar} Y_{\ell,\ell}$$

$$= \frac{e^{-i\phi}}{\sqrt{2\ell}} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right) C (\sin\theta)^{\ell} e^{i\ell\phi}$$

$$= \frac{e^{-i\phi} C}{\sqrt{2\ell}} \left(-\ell (\sin\theta)^{\ell-1} \cos\theta - \ell (\cos\theta)(\sin\theta)^{\ell-1} \right) e^{i\ell\phi}$$

$$Y_{\ell,\ell-1} = -\sqrt{2\ell}\, C\, (\sin\theta)^{\ell-1} (\cos\theta)\, e^{i(\ell-1)\phi}$$

etc

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$$\int |Y_{lm}|^2 \sin\theta \, d\theta \, d\phi = 1$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$ $l = 0$ S state (sharp)

~~Convention~~

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$$

$l = 1$ P state (principle)

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\phi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$Y_{2,-1} = +\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{-i\phi}$$

$$Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{-2i\phi}$$

$l = 2$ D state (diffuse

spdfgh

fundamental)

$$\boxed{Y_{l,-m} = (-1)^m Y^*_{l,m}}$$

avoid this

Griffiths (2e) 139

$$\left[Y_{lm} = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi \, (l+|m|)!}} P_l^{|m|}(\cos\theta) \, e^{im\phi} \qquad \epsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases} \right]$$