3d Potential w/ spherical symmetry

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \qquad r = \sqrt{x_1^2+x_2^2+x_3^2}$$

H is invariant under <u>rotations</u>

Rotations generated by angular momentum $\vec{J}$

$$U = e^{-\frac{i\vec{\omega}\cdot\vec{J}}{\hbar}}$$ acts on states $|\psi\rangle$

Let $H|E\rangle = E|E\rangle$

Then $U|E\rangle$ is also an eigenstate w/ same E.
(because H is unchanged under rotation)

$$H\,U|E\rangle = E\,U|E\rangle = U\,E|E\rangle = U H|E\rangle$$

ie $[H, U]|E\rangle = 0$

Since energy eigenstates are complete

$\Rightarrow [H, U]|\psi\rangle = 0$ for any $|\psi\rangle$

$\Rightarrow [H, U] = 0$

$$U = 1 - \frac{i\vec{\omega}\cdot\vec{J}}{\hbar} + O(\omega^2)$$

$\Rightarrow [H, \vec{\omega}\cdot\vec{J}] = 0$

$\Rightarrow [H, J_i] = 0$

Hamiltonian of a central potential is compatible w/ all components of angular momentum

$\Rightarrow$ find mutual eigenstates

Ignore spin for now: $\vec{J} = \vec{L} = \vec{X} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$

$L_3 = x_1 p_2 - x_2 p_1$ etc.

Earlier we asserted
$$[J_1, J_2] = i\hbar J_3$$
$$[J_2, J_3] = i\hbar J_1$$
$$[J_3, J_1] = i\hbar J_2$$

[HW] Show, using $[x_i, p_j] = i\hbar \delta_{ij}$ that orbital angular momentum operators L_i obey the angular momentum commutation relations

Observe that in position space

$$L_3 = \frac{\hbar}{i}\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

Proof: $$\frac{\partial}{\partial \phi} = \frac{\partial x_1}{\partial \phi}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \phi}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial \phi}\frac{\partial}{\partial x_3}$$

$$= \underbrace{-r\sin\theta\sin\phi}_{x_2}\frac{\partial}{\partial x_1} + \underbrace{r\sin\theta\cos\phi}_{x_1}\frac{\partial}{\partial x_2} \quad \checkmark$$

[HW] Show that

$$L_1 = i\hbar\left(\sin\phi \frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi}\right)$$

$$L_2 = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

Consider rotation through ω about z-axis

$$U = e^{-\frac{i\omega L_3}{\hbar}} = e^{-\omega\frac{\partial}{\partial\phi}}$$

$$U\psi(r,\theta,\phi) = \left(1 - \omega\frac{\partial}{\partial\phi} + \frac{1}{2}\omega^2\frac{\partial^2}{\partial\phi^2} + \dots\right)\psi(r,\theta,\phi)$$

$$= \psi(r,\theta,\phi-\omega) \quad \checkmark$$

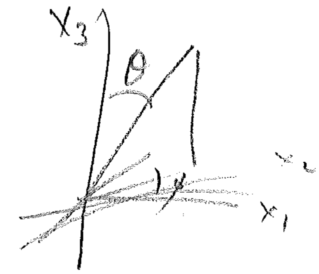
Earlier, we argued that $[H, L_i] = 0$ (ignoring spin)

Let's check it explicitly for $L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$[L_3, V(r)] = 0$$

In spherical coordinates $\begin{cases} x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\sin\phi \\ x_3 = r\cos\theta \end{cases}$



$$\nabla^2 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)}_{\text{ie } \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}} + \frac{1}{r^2}\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)}_{\text{ie } \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Easy to see that $[\nabla^2, L_3] = 0$

$[\nabla^2, L_1] = 0$ and $[\nabla^2, L_2] = 0$ are not so easy to see

[But if write $[\nabla^2, L_3]$ in Cartesian coordinates & then let $1 \to 2 \to 3 \to 1$]

$$\Rightarrow [H, L_i] = 0$$

$[L_i, L_j] \neq 0$ so diff comps of orb. ang mom not compatible

However $[L^2, L_i] = 0$ where $L^2 = L_1^2 + L_2^2 + L_3^2$

[proved earlier in problem set]

Also $[H, L^2] = \sum_i [H, L_i^2] = \sum([H, L_i]L_i + L_i[H, L_i]) = 0$

$\Rightarrow$ H, L^2, and one component of $\vec{L}$ are mutually compatible

Choose $L_3 \Rightarrow$ $\{H, L^2, L_3\}$ = set of mutually commuting ops.

Goal: find simultaneous eigenstates $|E, l, m\rangle$

$$H|E, l, m\rangle = E\,|E, l, m\rangle$$
$$L^2|E, l, m\rangle = \hbar^2 l(l+1)\,|E, l, m\rangle$$
$$L_3\,|E, l, m\rangle = \hbar m\,|E, l, m\rangle$$