3 dimensions

observable $\hat{\vec{X}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$

different components are <u>compatible</u> $[\hat{x}_i, \hat{x}_j] = 0 \quad \forall i, j$

$\exists$ set of mutual eigenstates $|x_1, x_2, x_3\rangle$

$$\hat{x}_i |x_1, x_2, x_3\rangle = x_i |x_1, x_2, x_3\rangle, \quad \forall i$$

$$x_i \in \mathbb{R}$$

$$\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$$

orthonormality $\langle x_1', x_2', x_3' | x_1, x_2, x_3\rangle = \underbrace{\delta(x_1 - x_1')\,\delta(x_2 - x_2')\,\delta(x_3 - x_3')}$

$$\equiv \delta^{(3)}(\vec{x} - \vec{x}')$$

$$\int f(\vec{x})\,\delta^{(3)}(\vec{x} - \vec{x}')\,d^3\vec{x} = f(\vec{x}')$$

$$|\vec{x}\rangle \equiv |x_1, x_2, x_3\rangle \quad \text{so} \quad \langle \vec{x}' | \vec{x}\rangle = \delta^{(3)}(\vec{x} - \vec{x}')$$

completeness

$$\int d^3\vec{x}\; |\vec{x}\rangle\langle\vec{x}| = \mathbb{1}$$

arbitrary state

$$|\psi\rangle = \int d^3\vec{x}\; |\vec{x}\rangle \underbrace{\langle\vec{x}|\psi\rangle}_{\psi(\vec{x})} \qquad \psi(\vec{x}) = \text{wavefn in 3d position space}$$

normalization

$$1 = \langle\psi|\psi\rangle = \int d^3\vec{x}\; \langle\psi|\vec{x}\rangle\langle\vec{x}|\psi\rangle = \int d^3\vec{x}\; \psi^*(\vec{x})\,\psi(\vec{x}) = \int d^3x\; |\psi(\vec{x})|^3$$

$\hat{x}_i$ acts in position space as ~~multiplication~~ multiplication by x_i

$$\hat{x}_i|\psi\rangle = \int d^3\vec{x}\; \hat{x}_i|\vec{x}\rangle\,\psi(\vec{x}) = \int d^3\vec{x}\; x_i|\vec{x}\rangle\,\psi(x) = \int d^3x\; x_i\psi(x)|x\rangle$$

momentum = generator of spatial translations

$$U = e^{-\frac{i\vec{a}\cdot\vec{P}}{\hbar}}$$

$$U\psi(\vec{x}) = \psi(\vec{x}-\vec{a})$$

$$= e^{-\vec{a}\cdot\vec{\nabla}}\psi(x)$$

$\Rightarrow \quad \vec{P} = \frac{\hbar}{i}\vec{\nabla}$ on a position space wavefunction

$$p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i} \qquad i=1,2,3$$

$[\hat{p}_i, \hat{p}_j] = 0$ since $\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}$

formal

$|\vec{p}\rangle = |p_1, p_2, p_3\rangle$ mutual eigenstate of p_i

$$\hat{p}_i|\vec{p}\rangle = p_i|\vec{p}\rangle$$

$$\langle\vec{p}'|\vec{p}\rangle = \delta^{(3)}(\vec{p}-\vec{p}')$$

$$\int d^3p\, |\vec{p}\rangle\langle\vec{p}| = \mathbb{1}$$

$|\psi\rangle = \int d\vec{p}\, |\vec{p}\rangle \underbrace{\langle\vec{p}|\psi\rangle}_{\phi(\vec{p})}$ ← mom. space wavefunction

~~Heisenb~~

$$[x_i, p_j] = i\hbar\,\delta_{ij}$$

$$\left\{ [x, p_y] = x\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)x = 0 \right\}$$

Momentum space eigenfunction in position space

optional

$$\langle \vec{x} | \vec{p} \rangle = v_{\vec{p}}(\vec{x})$$

$$p_i v_{\vec{p}} = p_i v_{\vec{p}}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x_i} v_{\vec{p}} = p_i v_{\vec{p}}$$

$$\frac{dv_p}{v_p} = \frac{i p_i\, dx_i}{\hbar}$$

$$\ln v_p = \frac{i p_1 x_1}{\hbar} + f(x_2, x_3)$$

$$= \frac{i p_2 x_2}{\hbar} + f(x_1, x_3)$$

$$= \frac{i p_3 x_3}{\hbar} + f(x_1, x_2)$$

$$\text{ie} = \sum_i \frac{i}{\hbar} p_i x_i + \text{const}$$

$$= \frac{i}{\hbar}\vec{p}\cdot\vec{x} + \text{const}$$

$$\Rightarrow v_{\vec{p}} = e^{\sum_{i=1}^{3} \frac{i p_i x_i}{\hbar}} = e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$$

Dirac normalization $\quad V_{\vec{p}} = \dfrac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$

$$|\vec{p}\rangle = \int dx\, |\vec{x}\rangle\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}} |\vec{x}\rangle$$

$$|\vec{x}\rangle = \int dp\, |\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p\, e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} |\vec{p}\rangle$$

(unitary tx between bases.) $\qquad$ Fourier transform