Time evolution of a particle in a potential

$$|\psi(t)\rangle \quad \Rightarrow \quad \psi(x,t) = \langle x|\psi(t)\rangle$$

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \quad \Rightarrow \quad i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

time-dependent Schrodinger eqn

Decompose initial state into energy eigenstates $|E_n\rangle \Rightarrow u_n(x) = \langle x|E_n\rangle$

$$|\psi(0)\rangle = \sum c_n |E_n\rangle \quad \Rightarrow \quad \psi(x,0) = \sum c_n u_n(x)$$

$$c_n = \langle E_n|\psi(0)\rangle = \int dx \langle E_n|x\rangle\langle x|\psi(0)\rangle = \int dx\, u_n^*(x)\, \psi(x,0)$$

Then

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle = \sum_n c_n e^{\frac{iHt}{\hbar}}|E_n\rangle = \sum c_n e^{-\frac{iE_nt}{\hbar}}|E_n\rangle$$

ie $\quad \psi(x,t) = \sum c_n e^{-\frac{iE_nt}{\hbar}} u_n(x)$

If $\psi(x,0)$ consists of a single energy eigenstate

eg $\psi(x,0) = u_n(x)$

then $\psi(x,t) = u_n(x) e^{-\frac{iE_n t}{\hbar}}$ [separable solution of t.d.S.e.]

and $|\psi|^2 = |u_n(x)|^2$ is indep of time $\Rightarrow$ stationary state

all observables for this state do not depend on time

If $\psi(x,0)$ consists of <u>several</u> eigenstates (w/ different energies)

$\psi(x,0) = \sum c_n u_n(x)$

then $\psi(x,t) = \sum c_n u_n(x) e^{-\frac{iE_n t}{\hbar}}$ is not a stationary state

$$|\psi(x,t)|^2 = \left(\sum c_n^* u_n^*(x) e^{+\frac{iE_n t}{\hbar}}\right)\left(\sum c_m u_m(x) e^{-\frac{iE_m t}{\hbar}}\right)$$

$$= \underbrace{\sum_n |c_n|^2 |u_n(x)|^2}_{\text{indep of time}} + \sum_{n \neq m} c_n^* c_m \underbrace{u_n^*(x) u_m(x) e^{\frac{i(E_n - E_m)t}{\hbar}}}_{\text{cross-terms carry time dependence}}$$

(change / motion requires interference)

[unless degenerate]

Heisenberg eqn: $\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A,H]\rangle$

Consider $[\hat{p},\hat{H}] = [\hat{p}, V(\hat{x})]$

$$= [\hat{p}, V_0 + V_1\hat{x} + V_2\hat{x}^2 + \dots + V_n\hat{x}^n + \dots]$$

$$= V_1\underbrace{[\hat{p},\hat{x}]}_{-i\hbar} + V_2\underbrace{[\hat{p},\hat{x}^2]}_{[p,x]x + x[p,x] = -2i\hbar\hat{x}} + \dots + V_n\underbrace{[\hat{p},\hat{x}^n]}_{-n i\hbar\hat{x}^{n-1}} + \dots$$

$$= -i\hbar(V_1 + 2V_2\hat{x} + \dots + nV_n\hat{x}^{n-1})$$

$$= -i\hbar\frac{dV}{dx}(\hat{x})$$

Act on $\psi(x)$

$\hat{p}V(\hat{x})\psi = \frac{\hbar}{i}\frac{d}{dx}[V(x)\psi(x)]$

$V(\hat{x})\hat{p}\psi = V(x)\frac{\hbar}{i}\frac{d\psi}{dx}$

$\Rightarrow [\hat{p},\hat{V}]\psi = \left(\frac{\hbar}{i}\frac{dV}{dx}\right)\psi$

$$\Rightarrow \frac{d}{dt}\langle p\rangle = \left\langle -\frac{dV}{dx}\right\rangle$$

Recall classically $\frac{dp}{dt} = F_x = -\frac{dV}{dx}$

quantum analog replaces variables by expectation values

[Problem: $\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}$ ie $\langle p\rangle = m\frac{d}{dt}\langle x\rangle$

Sol: $[x,H] = [x,\frac{p^2}{2m}] = i\hbar\frac{p}{m}$]