Particle in a potential

$$E = K + V = \frac{p^2}{2m} + V(x)$$

$$\Rightarrow \quad \hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})$$

[what is meaning of $\hat{V}$?]

If $V(x) = V_0 + V_1 x + V_2 x^2 + \ldots$

then $\hat{V}(\hat{x}) = V_0 + V_1 \hat{x} + V_2 \hat{x}^2 + \ldots$

$\hat{H}$ is hermitian if $V(x)$ is a real function, ie if $V_0, V_1, \ldots$ are real

$$\hat{V}^\dagger = V_0^* + V_1^* \hat{x}^\dagger + V_2^* (\hat{x}^2)^\dagger + \ldots = \hat{V}$$

Energy eigenstates

$$\hat{H} |E\rangle = E |E\rangle$$

$$\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right] |E\rangle = E |E\rangle$$

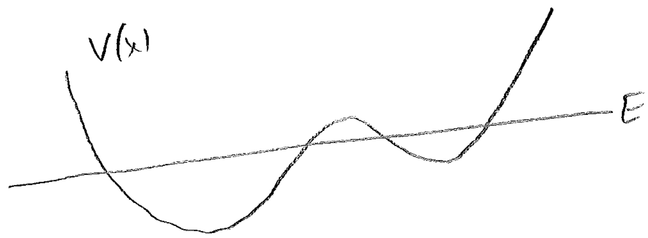
Energy eigenfunction in position space $u_E(x) = \langle x | E \rangle$

$$\hat{H}_{pos.} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m} \frac{d^2 u_E}{dx^2} + V(x)\, u_E(x) = E\, u_E(x)$$

time-independent Schrodinger eqn

$$\Rightarrow \quad \frac{1}{u_E}\frac{d^2 u_E}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V(x)\right)$$

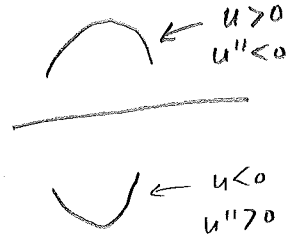


classically allowed regions $\Rightarrow E - V(x) > 0 \Rightarrow \frac{u''}{u} < 0 \Rightarrow$ concave toward x-axis

classically forbidden regions $\Rightarrow E - V(x) < 0 \Rightarrow \frac{u''}{u} > 0 \Rightarrow$ concave away from axis

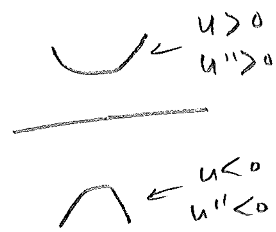
turning point $\Rightarrow E - V(x) = 0 \Rightarrow \frac{u''}{u} = 0 \Rightarrow$ point of inflection

concave toward axis



$\frac{u''}{u} < 0$

concave away from axis



$\frac{u''}{u} > 0$

Boundary conditions on wavefunction $u(x)$

① $u(x) \longrightarrow 0$ as $x \to \pm\infty$

② $u(x)$ continuous at all points

③ $\frac{du}{dx}(x)$ continuous, except if $V(x)$ is infinite

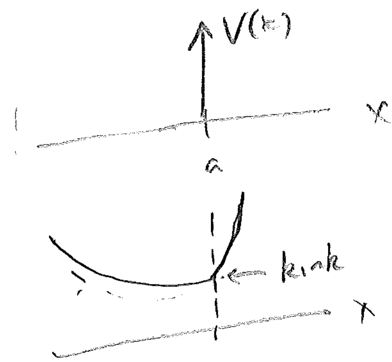
Consider $$\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u}{dx^2}\,dx = \int_{a-\epsilon}^{a+\epsilon} \frac{d}{dx}\left(\frac{du}{dx}\right) dx = \frac{du}{dx}\Big|_{a-\epsilon}^{a+\epsilon}$$

But $$\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u}{dx^2}\,dx = -\frac{2m}{\hbar^2}\int_{a-\epsilon}^{a+\epsilon} (E - V(x))\,u(x)\,dx$$

If $V(x)$, $u(x)$ finite, the integral $\propto \epsilon$ $\Rightarrow \frac{du}{dx}(a+\epsilon) - \frac{du}{dx}(a-\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0$

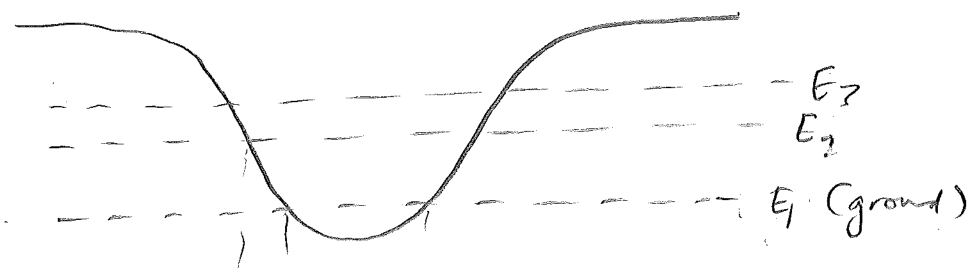
Suppose $V(x) = +\alpha\,\delta(x-a)$ then $-\frac{2m}{\hbar^2}\int (E - \alpha\,\delta(x-a))\,u(x)\,dx$

$$\frac{du}{dx}(a+\epsilon) - \frac{du}{dx}(a-\epsilon) = \frac{2m\alpha}{\hbar^2}\,u(a)$$

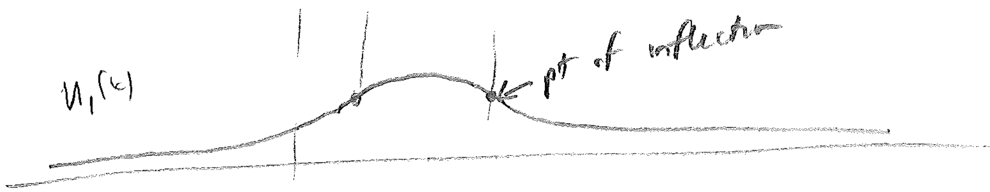


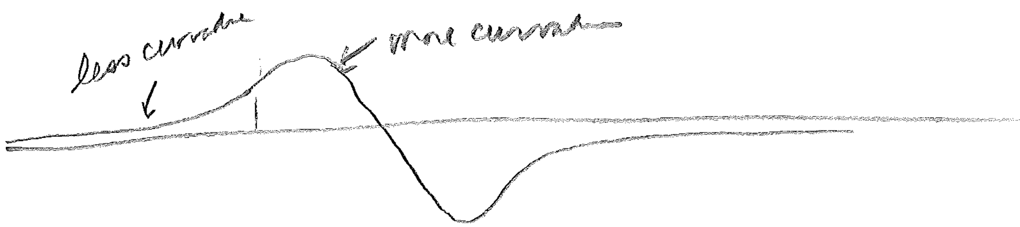
Recall: ~~[illegible]~~ square integrable ⇒ $u \to 0$ as $x \to \pm\infty$

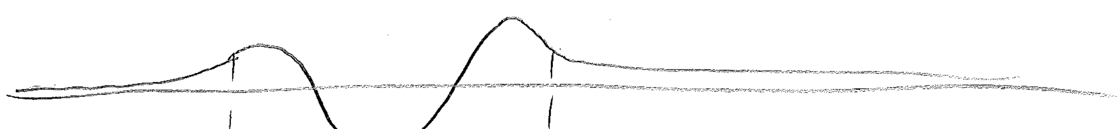
e.g.



$u(x) \xrightarrow[x \to \pm\infty]{} 0$ in forbidden region





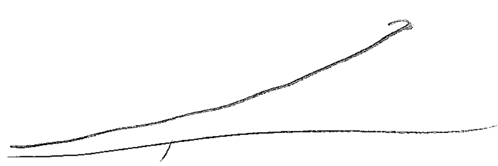


$E \uparrow$, more wiggly ⇒ more nodes

Suppose $E < V_{min}$ ⇒ all regions forbidden

⇒ always concave away

⇒ not normalizable



⇒ no energy eigenvalues w/ $E < V_{min}$!

If the potential is even : $V(x) = V(-x)$

then nondegenerate eigenfunctions are either even or odd

Prop: Let $u(x)$ obey

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + V(x)\,u(x) = E\,u(x)$$

Let $x \to -x$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + V(-x)\,u(-x) = E\,u(-x)$$

$$\downarrow$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + V(x)\,u(-x) = E\,u(-x)$$

ie $u(-x)$ obeys the same eqn as does $u(x)$

Since $u(x)$ is nondegenerate, $u(-x)$ must be prop. to $u(x)$

$$u(-x) = \lambda u(x)$$

Let $x \to -x$

$$u(x) = \lambda u(-x) = \lambda^2 u(x) \Rightarrow \lambda^2 = 1$$

$$\lambda = \pm 1$$

$\lambda = 1$: $u(-x) = u(x)$ even

$\lambda = -1$ $u(-x) = -u(x)$ odd

Recall: ptcl in box