Free particle in 1 dimension

Free particle: all energy is kinetic

$$E = K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$\hat{H}$ hermitian because $\hat{p}$ hermitian

momentum eigenstates are also energy eigenstates

$$\hat{H}|p\rangle = \frac{\hat{p}^2}{2m}|p\rangle = \frac{p^2}{2m}|p\rangle \quad \Rightarrow \quad E_p = \frac{p^2}{2m}$$

$|\psi\rangle = \int dp \; \phi(p)\,|p\rangle$ is the expansion of $|\psi\rangle$ into energy eigenstates

NB. $|p\rangle$ and $|-p\rangle$ are degenerate w.r.t. $\hat{H}$.

Recall dual role of $\hat{H}$: energy operator
and generator of time translations

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$$

$$= \int dp\, \phi(p)\, e^{-\frac{i\hat{H}t}{\hbar}}|p\rangle$$

$$= \int dp\, \underbrace{\phi(p)\, e^{-i\frac{p^2 t}{2m\hbar}}}_{\phi(p,t)}|p\rangle$$

A momentum space wavefunction $\phi(p)$ at $t=0$
evolves to $\phi(p)\, e^{-\frac{ip^2t}{2m\hbar}}$ at time t

$\Rightarrow |\phi(p,t)|^2 = |\phi(p)|^2 \Rightarrow$ prob. density in mom. space doesn't change

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Position space wavefunction evolves to

$$\psi(x,t) = \langle x|\psi(t)\rangle = \int dp\, \phi(p)\, e^{-\frac{ip^2t}{2m\hbar}}\langle x|p\rangle$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int dp\, \phi(p)\, \underbrace{e^{i\left(\frac{p}{\hbar}x - \frac{p^2}{2m\hbar}t\right)}}_{\text{travelling wave}}}_{\text{linear combination of travelling waves}}$$

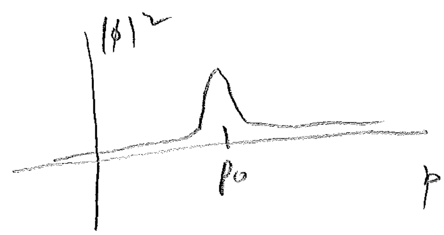
$\equiv$ wave packet

[www.eng.fsu.edu/~dommelen/ publications html/3140 for demo of gp vs phase velocity]

Group velocity = speed of wave packet $= \frac{d\omega}{dk} = \frac{p}{m} = v_{classical}$

Consider a gaussian momentum distribution

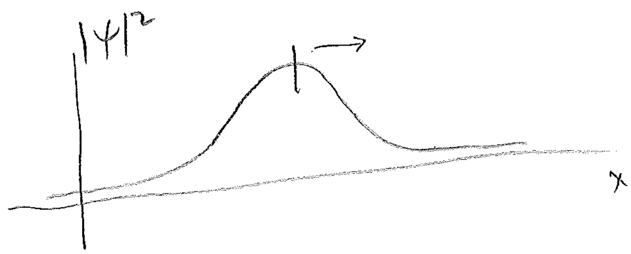
$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$



Position space wavefunction evolves to

$$|\psi(x,t)|^2 = \left(\frac{1}{2\pi f^2(t)}\right)^{\frac{1}{2}} e^{-\frac{(x-\frac{p_0}{m}t)^2}{2f^2(t)}}$$

You will calc $f(t)$ in HW, but $f(t)$ grows monotonically w/ t



$\langle x\rangle = \frac{p_0}{m}t$

Peak moves w/ speed $\frac{p_0}{m}$. $\Rightarrow$ not a stationary state because of interference btw energy eigenstates

gaussian spreads out w/ time (position becomes more uncertain)

[HW: derive ~~show~~ $\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}$; show that it is satisfied here.]

QM. w/ Mathematica