Momentum space wavefunction

Use completeness $\int dp\, |p\rangle\langle p| = 1$ to write

$$|\psi\rangle = \int dp\ |p\rangle\ \langle p|\psi\rangle$$

Define $\phi(p) = \langle p|\psi\rangle$ = wavefunct of ψ in momentum space
prob. amp.

$$|\psi\rangle = \int dp\ \phi(p)\ |p\rangle$$

$$1 = \langle\psi|\psi\rangle = \int dp\, \langle\psi|p\rangle\langle p|\psi\rangle = \int dp\ |\phi(p)|^2$$

ϕ square integrable in p-space.

$|\phi(p)|^2$ = probability density in momentum space

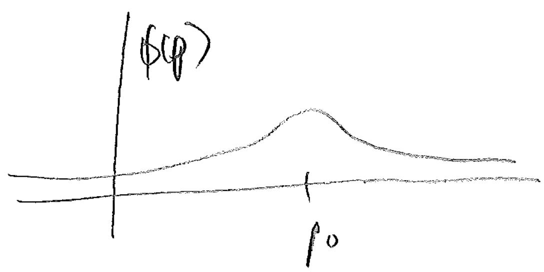
$$\langle p\rangle = \langle\psi|\hat{P}|\psi\rangle = \int dp\, \langle\psi|\ \underbrace{\hat{P}\ |p\rangle}_{p|p\rangle}\ \langle p|\psi\rangle = \int_{-\infty}^{\infty} dp\ |\phi(p)|^2 p$$

[$|p\rangle$ is an idealization; perfectly well-defined p] → ∞'ly spread out in position space
try for something more realistic: range of momenta]

29-2

gaussian

Consider a particle of momentum space distribution $\phi(p) = N e^{-\beta(p-p_0)^2}$



[expect to measure p_0, of some spread]

$$1 = \int dp\, |\phi(p)|^2 = N^2 \int_{-\infty}^{\infty} dp\, e^{-2\beta(p-p_0)^2} = N^2 \underbrace{\int_{-\infty}^{\infty} dp'\, e^{-2\beta(p')^2}}_{I(\beta)}$$

$$\Rightarrow N = \frac{1}{\sqrt{I(\beta)}}$$

$$I = \int_{-\infty}^{\infty} dx\, e^{-2\beta x^2}$$

$$I^2 = \int_{-\infty}^{\infty} dx\, e^{-2\beta x^2} \int_{-\infty}^{\infty} dy\, e^{-2\beta y^2} = \int \underbrace{dx\, dy}_{r\, dr\, d\theta}\, e^{-2\beta \overbrace{(x^2+y^2)}^{r^2}}$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\infty} r e^{-2\beta r^2} dr}_{\frac{1}{4\beta}\underbrace{\int_0^{\infty} du\, e^{-u}}_{1}}$$

$u = 2\beta r^2$
$du = 4\beta r\, dr$

$$= \frac{\pi}{2\beta}$$

$$I = \sqrt{\frac{\pi}{2\beta}} \quad \Rightarrow \quad \phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$

$$\langle p \rangle = \int dp\, |\phi(p)|^2 p$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp\, e^{-2\beta(p-p_0)^2} p \qquad \text{Let } p = p_0 + p'$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp'\, e^{-2\beta p'^2} (p_0 + p')$$

$$= p_0 \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp'\, e^{-2\beta p'^2} + \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} dp'\, e^{-2\beta p'^2} p'}_{\text{odd integrand}}$$

$$= p_0$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} dp\, |\phi(p)|^2 p^2$$

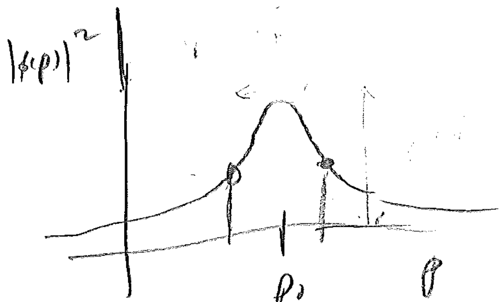
$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp'\, e^{-2\beta p'^2} (p_0^2 + 2p_0 p' + p'^2)$$

$$= p_0^2 + 0 + \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} dp'\, e^{-2\beta p'^2} p'^2}$$

$$I(\beta) = \int dp'\, e^{-2\beta p'^2} = \sqrt{\frac{\pi}{2\beta}} \longrightarrow -\frac{1}{2}\frac{dI}{d\beta} = \frac{1}{4}\sqrt{\frac{\pi}{2}}\,\frac{1}{\beta^{3/2}}$$

$$\langle p^2 \rangle = p_0^2 + \frac{1}{4\beta}$$

[problem: $\langle p^{2n} \rangle$]

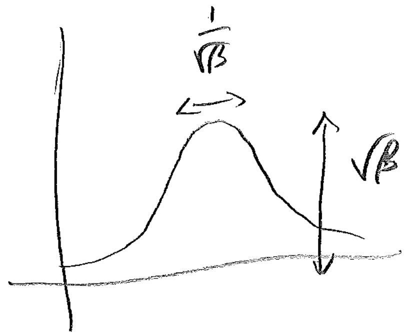


$$|f(p)|^2 = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} e^{-2\beta(p-p_0)^2}$$

$$= e^{-1}$$

$$2\beta\,(p-p_0)^2 = 1$$

$$|p-p_0| \simeq \frac{1}{\sqrt{2\beta}}$$



more precise: $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{1}{2\sqrt{\beta}}$

$$|\psi\rangle = \int dx \ \psi(x) \ |x\rangle$$
$$= \int dp \ \phi(p) \ |p\rangle$$

How are $\phi(p)$ + $\psi(x)$ related? Fourier transforms!

$$\psi(x) = \langle x|\psi\rangle = \int dp \ \phi(p) \ \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp \ \phi(p) \ e^{\frac{ipx}{\hbar}}$$

$\psi(x)$ = superposition of plane waves weighted by $\phi(p)$

$$\phi(p) = \langle p|\psi\rangle = \int dx \ \psi(x) \ \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \ \psi(x) \ e^{-\frac{ipx}{\hbar}}$$

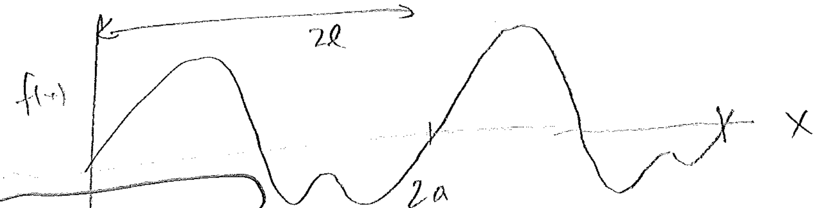
Quick review

these are part (a) + (b) Griffiths problem

I then assign part (c) + (d)

ES

Periodic function $f(x+2a) = f(x)$

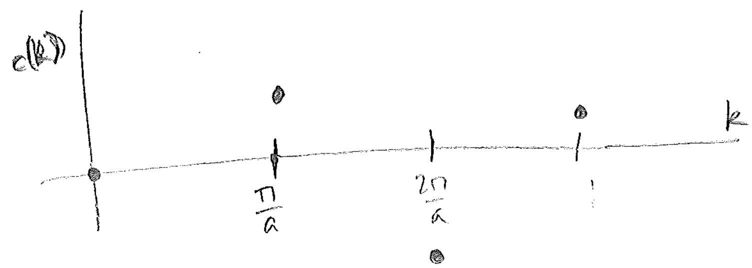


Recall from Phys 360 Then $f(x) = \sum (a_n \quad + b_n \quad)$ ← cf Griffiths (n.b. a, b complex since f complex)

Then $f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{i\pi n x}{a}} = \sum_{k \in \frac{\pi}{a}\mathbb{Z}} c(k) e^{ikx}$ [Fourier sum]

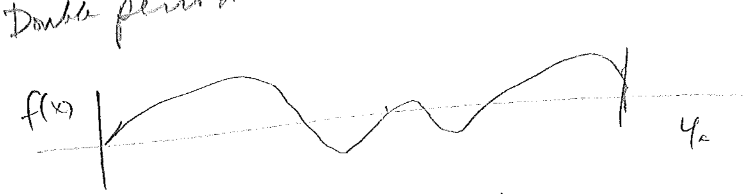
let $k = \frac{\pi n}{a}$

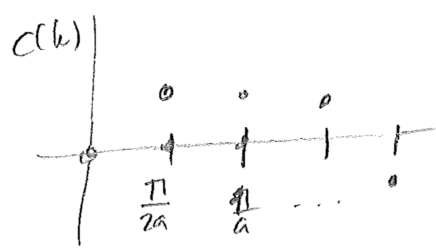
Same information



[2nd plot is discrete but goes to ∞; first is continuous but limited to $0 \le x \le 2a$]

Double period

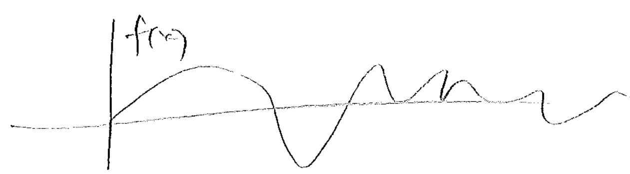


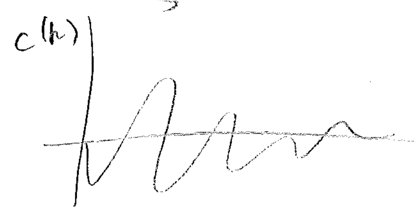


twice as much info in each.

Let $a \to \infty$

Fourier transform





Can calc $\langle p \rangle$ using either $\phi(p)$ or $\psi(x)$

$$\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = \int dp \; \phi^*(p) \, p \, \phi(p)$$

Now Fourier transform

$$= \int dp \; \phi^*(p) \, p \, \frac{1}{\sqrt{2\pi\hbar}} \int dx \; \psi(x) e^{-\frac{ipx}{\hbar}}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp \; \phi^*(p) \underbrace{\int_{-\infty}^{\infty} dx \; \psi(x) \underbrace{p \, e^{-\frac{ipx}{\hbar}}}_{-\frac{\hbar}{i}\frac{d}{dx} e^{-\frac{ipx}{\hbar}}}}_{}$$

$$\int u \frac{dv}{dx} = \underbrace{\int dx \frac{d}{dx}(uv)}_{uv|} - \int dx \frac{du}{dx} v$$

$$\underbrace{-\frac{\hbar}{i} \psi(x) e^{-\frac{ipx}{\hbar}} \Big|_{-\infty}^{\infty}}_{0} - \int dx \left(-\frac{\hbar}{i}\right) \frac{d\psi}{dx} e^{-\frac{ipx}{\hbar}}$$

$$= \int_{-\infty} dx \quad \frac{\hbar}{i} \frac{d\psi}{dx} \; \underbrace{\frac{1}{\sqrt{2\pi\hbar}} \int dp \; \phi^*(p) e^{-\frac{ipx}{\hbar}}}_{\psi^*(x)}$$

$$= \int_{-\infty}^{\infty} dx \quad \psi^*(x) \left(\frac{\hbar}{i} \frac{d}{dx}\right) \psi(x)$$

$$\langle A \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x) \; \hat{A}_{pos.} \psi(x)$$

in position space $\begin{cases} \hat{x}_{pos.} & x \\ \hat{p}_{pos} & \frac{\hbar}{i} \frac{d}{dx} \end{cases}$

(Prob: show that $\hat{p}$ is hermitian)

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{\frac{ipx}{\hbar}} \phi(p)$$

what is the position space wavefunction of a gaussian momentum distrib.?

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2} \quad \Rightarrow \Delta p = \frac{1}{2\sqrt{\beta}}$$

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}}\left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} \underbrace{\int dp \; e^{-\beta(p-p_0)^2 + \frac{ipx}{\hbar}}}_{e^{\frac{ip_0x}{\hbar}} \int dq \; e^{-\beta q^2 + \frac{iqx}{\hbar}}}$$

Let $q = p - p_0 \Rightarrow p = q + p_0$

Consider $\int dq \; e^{-aq^2 + bq}$

$$= \int dq \; e^{-a\left(q - \frac{b}{2a}\right)^2 + \frac{b^2}{4a}}$$

$$= e^{\frac{b^2}{4a}} \int dq' \; e^{-aq'^2}$$

$$= \sqrt{\frac{\pi}{a}} \, e^{b^2/4a} \checkmark$$

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}}\left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}}\left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} e^{\frac{1}{4\beta}\left(\frac{ix}{\hbar}\right)^2} e^{\frac{ip_0x}{\hbar}}$$

$$= \left(\frac{1}{2\pi\hbar^2\beta}\right)^{\frac{1}{4}} e^{-\frac{x^2}{4\beta\hbar^2}} e^{\frac{ip_0x}{\hbar}}$$

$$= \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} e^{\frac{ip_0x}{\hbar}} \quad \text{where } \alpha = \frac{1}{4\beta\hbar^2}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{\frac{ipx}{\hbar}} \phi(p)$$

Alternative calculation, but not as suitable for generalization to time dependent case → see next page

What is the position space wavefunction of a gaussian momentum distribution?

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2} \qquad \Delta p = \frac{1}{2\sqrt{\beta}}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} \int dp \; e^{-\beta p^2 + 2\beta p_0 p - \beta p_0^2 + \frac{ipx}{\hbar}}$$

Consider $\int_{-\infty}^{\infty} dp \; e^{-ap^2 + bp + c}$

$$= \int_{-\infty}^{\infty} dp \; e^{-a\left(p - \frac{b}{2a}\right)^2 + \frac{b^2}{4a} + c}$$

$$= e^{\frac{b^2}{4a} + c} \int_{-\infty}^{\infty} dp' \; e^{-a(p')^2}$$

$$= e^{\frac{b^2}{4a} + c} \sqrt{\frac{\pi}{a}}$$

Then

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} e^{\frac{1}{4\beta}\left(\frac{ix}{\hbar} + 2\beta p_0\right)^2 - \beta p_0^2}$$

$$= \left(\frac{1}{2\pi\hbar^2\beta}\right)^{\frac{1}{4}} e^{-\frac{x^2}{4\beta\hbar^2} + i\frac{p_0 x}{\hbar}}$$

$$= \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} e^{\frac{ip_0 x}{\hbar}} \qquad \text{where } \alpha = \frac{1}{4\beta\hbar^2}$$

Trying to do the travelling gaussian w/o doing the shift.
(this is more subtle + not suitable for thw)

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2\beta}{\pi}\right)^{\frac14} \underbrace{\int dp\; e^{-\beta(p-p_0)^2 + \frac{ipx}{\hbar} - \frac{ip^2 t}{2m\hbar}}}$$

$$e^{-\beta p_0^2} \underbrace{\int dp\; e^{-[\underbrace{\beta + \frac{it}{2m\hbar}}_{a}]p^2 + [\underbrace{\frac{ix}{\hbar} + 2\beta p_0}_{b}]p}}$$

$$\sqrt{\frac{\pi}{a}}\; e^{b^2/4a}$$

$$= \left(\frac{1}{2\pi\hbar}\right)^{\frac12}\left(\frac{2\beta}{\pi}\right)^{\frac14}\left(\frac{\pi}{\beta + \frac{it}{2m\hbar}}\right)^{\frac12} \underbrace{e^{-\beta p_0^2 + \frac{\left(\frac{ix}{\hbar} + 2\beta p_0\right)^2}{4\left(\beta + \frac{it}{2m\hbar}\right)}}}$$

$$e^{\frac{1}{4\left(\beta + \frac{it}{2m\hbar}\right)}\left[-\frac{x^2}{\hbar^2} + \frac{4\beta p_0 ix}{\hbar} + \underbrace{4\beta^2 p_0^2 - 4\beta p_0^2\left(\beta + \frac{it}{2m\hbar}\right)}_{-\frac{2\beta p_0^2 it}{m\hbar}}\right]}$$

this is not helpful

$$e^{\frac{1}{4\left(\beta + \frac{it}{2m\hbar}\right)}\left[-\frac{x^2}{\hbar^2} + \frac{4\beta p_0 i}{\hbar}\left(x - \frac{p_0}{2m}t\right)\right]}$$

$$e^{-\beta p_0^2 + \frac{\left(\frac{ix}{\hbar} + 2\beta p_0\right)^2\left(\beta - \frac{it}{2m\hbar}\right)}{4\left(\beta^2 + \frac{t^2}{4m^2\hbar^2}\right)}}$$

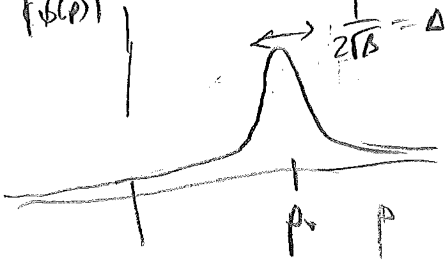
$$\underbrace{\left(\frac{ix}{\hbar} + 2\beta p_0\right)^2\left(\beta - \frac{it}{2m\hbar}\right)} = \beta\left(-\frac{x^2}{\hbar^2} + 4\beta^2 p_0^2\right) + \left(\frac{4\beta p_0 x}{\hbar}\right)\left(\frac{t}{2m\hbar}\right) + (\text{imag part})$$

$$= -\frac{\beta}{\hbar^2}\left(x - \frac{p_0}{m}t\right)^2 + \underbrace{\underbrace{\frac{\beta p_0^2 t^2}{m^2\hbar^2} + 4\beta^3 p_0^2}_{\beta p_0^2\left(4\beta^2 + \frac{t^2}{m^2\hbar^2}\right)}}_{\text{cancels off first piece}}$$

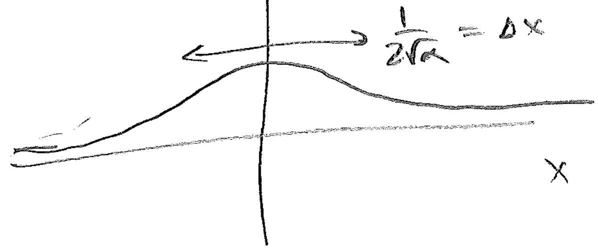
$$\text{so } e^{\frac{-\beta\left(x - \frac{p_0}{m}t\right)^2}{4\hbar^2\left(\beta^2 + \frac{t^2}{4m^2\hbar^2}\right)} + i(\text{stuff})}$$

Fourier tx of a gaussian is a gaussian (times a phase)

$$|\phi(p)|^2 = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} e^{-2\beta(p-p_0)^2} \qquad \frac{1}{2\sqrt{\beta}} = \Delta p$$



$$|\psi(x)|^2 = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} e^{-2\alpha x^2} \qquad \frac{1}{2\sqrt{\alpha}} = \Delta x$$

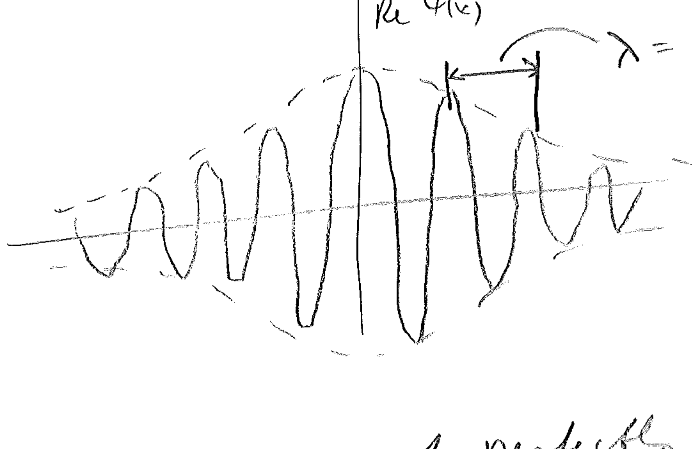


Since $\alpha\beta = \frac{1}{4\hbar^2}$

products of widths $\left(\frac{1}{2\sqrt{\alpha}}\right)\left(\frac{1}{2\sqrt{\beta}}\right) = \frac{\hbar}{2}$ widths are inverse related

$\Delta x \, \Delta p = \frac{\hbar}{2}$ Heisenberg is saturated

What about the phase? $\mathrm{Re}\,\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} \cos\left(\frac{p_0 x}{\hbar}\right)$



$\lambda = 2\pi\left(\frac{\hbar}{p_0}\right) = \frac{h}{p_0}$

= de Broglie wavelength of particle w/ mom p_0

[wavelength not perfectly well defined because momentum not " "]