Momentum eigenstates

[Besides being a generator of translations]

Momentum is an observable

$$\hat{P}|p\rangle = p|p\rangle$$

p = possible result of a measurement of momentum = any real #

$|p\rangle$ = state with perfectly well-defined momentum (idealization)

Completeness $\int dp\, |p\rangle\langle p| = \mathbb{1}$

orthonormality $\langle p'|p\rangle = \delta(p-p')$

Denote wave-funct of $|p\rangle$ by $\langle x|p\rangle = \psi_p(x)$

ie $|p\rangle = \int dx\, \psi_p(x)\, |x\rangle$

$$\frac{\hbar}{i}\frac{d}{dx}\psi_p(x) = p\,\psi_p$$

$$\frac{d\psi_p}{\psi_p} = \frac{ip}{\hbar}\,dx$$

$$\ln \psi_p = \frac{ipx}{\hbar} + \text{const}$$

$$\psi_p = A\, e^{\frac{ipx}{\hbar}}$$

$$\Rightarrow \langle x|p\rangle = A\, e^{\frac{ipx}{\hbar}}$$ (A still to be determined)

N.B. $|p\rangle$ not normalizable : $\int dx\, |\psi_p(x)|^2 = \int_{-\infty}^{\infty} dx\, |A|^2 = \infty$.

Also, equally likely to be anywhere in space!

$$\Delta x \geq \frac{\hbar}{2\Delta p} = \infty$$

$|p\rangle$ form an orthonormal basis in the Dirac sense

ie, $\langle p'|p\rangle = \delta(p'-p)$
Use this to determine A.

$$\delta(p-p') = \int_{-\infty}^{\infty} dx \, \langle p'|x\rangle\langle x|p\rangle$$

$$= \int_{-\infty}^{\infty} dx \; A^* e^{-\frac{ip'x}{\hbar}} \; A e^{\frac{ipx}{\hbar}}$$

$$= |A|^2 \int_{-\infty}^{\infty} dx \; e^{\frac{i(p-p')x}{\hbar}}$$

[Can "see" that r.h.s = 0 if $p \neq p'$ because cos + i sin = area vanishes.
But need to choose particular $|A|^2$ for it to be normalized

$$= |A|^2 \lim_{L\to\infty} \underbrace{\int_{-L}^{L} dx \; e^{\frac{i(p-p')x}{\hbar}}}_{\frac{\hbar}{i(p-p')} \underbrace{e^{\frac{i(p-p')x}{\hbar}} \Big|_{-L}^{L}}_{2i \sin\frac{(p-p')L}{\hbar}}}$$

$$\delta(p-p') = 2\hbar |A|^2 \lim_{L\to\infty} \frac{\sin\frac{(p-p')L}{\hbar}}{(p-p')}$$

Integrate both sides wrt. p

$$\int_{-\infty}^{\infty} \delta(p-p')\,dp = 2\hbar|A|^2 \lim_{L\to\infty} \underbrace{\int_{-\infty}^{\infty} dp \frac{1}{(p-p')} \sin\left[\frac{(p-p')L}{\hbar}\right]}$$

Let $y = \frac{(p-p')L}{\hbar}$ $$\underbrace{\int_{-\infty}^{\infty} dy \frac{\sin y}{y}}_{\pi}$$ — (see next page)

$$\underline{1} = 2\pi\hbar|A|^2$$

$$A = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\Rightarrow \boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}}$$

$$|p\rangle = \int dx\, |x\rangle\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{\frac{ipx}{\hbar}} |x\rangle$$

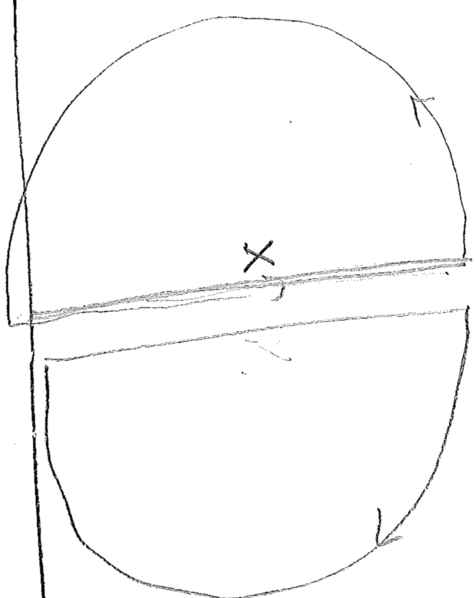
$$|x\rangle = \int dp\, |p\rangle\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, e^{-\frac{ipx}{\hbar}} |p\rangle$$

[change of basis]

↓

unitary transformation

$$\int_{-\infty}^{\infty} \frac{\sin y}{y}\, dy = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{\sin y}{y - i\epsilon} \qquad (\text{assume } \epsilon > 0)$$



$$= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{2i}\left(\frac{e^{iy} - e^{-iy}}{y - i\epsilon}\right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{2i} \underbrace{\oint_{UHP} \frac{e^{iy}}{y - i\epsilon}} - \frac{1}{2i} \underbrace{\oint_{LHP} \frac{e^{-iy}}{y - i\epsilon}}_{0}$$

Pole is at $y = i\epsilon$

$$2\pi i \,\text{Res}$$
$$= 2\pi i\, e^{-\epsilon}$$

$$= \pi \quad ✓$$