Backgrd

$$e^{-\frac{i\epsilon p}{\hbar}} X e^{\frac{i\epsilon p}{\hbar}} = X + \frac{i\epsilon}{\hbar}[X, P] = X - \epsilon$$

$$\langle x_0 | e^{-\frac{i\epsilon p}{\hbar}} X e^{\frac{i\epsilon p}{\hbar}} | x_0 \rangle = x_0 - \epsilon$$

$$\Rightarrow e^{\frac{i\epsilon p}{\hbar}} | x_0 \rangle = | x_0 - \epsilon \rangle$$

Now $|\psi\rangle = \int dx \, \psi(x) \, |x\rangle$

$$e^{\frac{i\epsilon p}{\hbar}} |\psi\rangle = \int dx \, \psi(x) \, e^{\frac{i\epsilon p}{\hbar}} |x\rangle$$

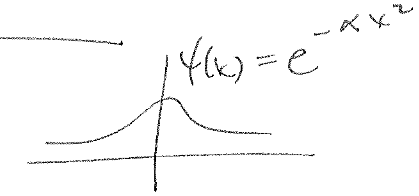
$$= \int dx \, \psi(x) \, |x - \epsilon\rangle$$

$$= \int dx \, \psi(x+\epsilon) \, |x\rangle$$

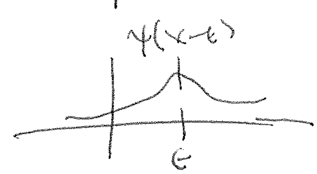
So acting on $\psi(x)$ we have

$$e^{i\epsilon \frac{p}{\hbar}} \psi(x) = \psi(x+\epsilon)$$

$$e^{\epsilon \frac{d}{dx}} \psi(x) = \psi(x+\epsilon)$$



$$e^{-i\epsilon \frac{p}{\hbar}} \psi(x) = \psi(x - \epsilon)$$



$$e^{-\frac{i\epsilon p}{\hbar}} |x\rangle = |x_0 + \epsilon\rangle$$

$U = e^{-\frac{i\epsilon p}{\hbar}}$ shifts state to the right by ϵ

Momentum operator

Recall that a spatial rotation $\vec{\omega}$ is implemented by unitary operator $U = e^{-\frac{i\vec{\omega}\cdot\vec{J}}{\hbar}}$ where $\vec{J}$ = ang mom operator

i.e. $\vec{J}$ generates spatial rotation

Recall that a time translation by t is implemented by unitary operator $U = e^{-\frac{itH}{\hbar}}$ where $\hat{H}$ = hamiltonian = energy operator

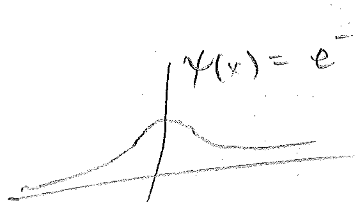
i.e. $\hat{H}$ generates time translation

A spatial translation by a is implemented by unitary op $U = e^{-\frac{ia\hat{P}}{\hbar}}$ where $\hat{P}$ = momentum operator

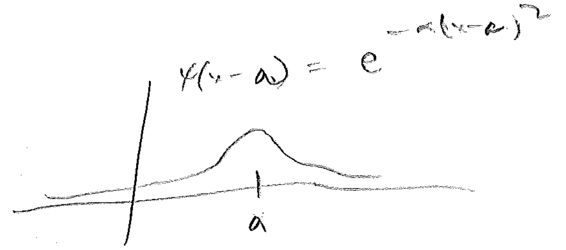
i.e. P generates spatial translations

U acts on a position-space wavefunction $\psi(x)$ to give $\psi(x-a)$

eg



$\Rightarrow$



$$U\psi(x) = \psi(x-a)$$

But $\psi(x-x_0) = \psi(x) - a\frac{d\psi}{dx} - \frac{1}{2!}a^2\frac{d^2\psi}{dx^2} + \dots$

$$= e^{-a\frac{d}{dx}}\psi(x)$$

So $-a\frac{d}{dx} = -\frac{i}{\hbar}a\hat{P} \Rightarrow \boxed{\hat{P} = \frac{\hbar}{i}\frac{d}{dx} \text{ on wavefunctions}}$

$\hat{X}$ acts on $\psi(x)$ to give $x\psi(x)$

$\hat{P}$ acts on $\psi(x)$ to give $\frac{\hbar}{i}\frac{d}{dx}\psi(x)$

~~$\hat{X}$ acts on $\psi(x)$ to give $x\psi(x)$~~

both are **linear** operators

Consider $[\hat{X},\hat{P}]\,\psi(x) = XP\psi - PX\psi$

$$= x\frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}\underbrace{\frac{d}{dx}(x\psi)}_{\psi + x\frac{d\psi}{dx}}$$

$$= i\hbar\,\psi(x)$$

Since this is true for arbitrary $\psi(x)$, we find

$$\boxed{[\hat{X},\hat{P}] = i\hbar\,\mathbb{1}}$$ Heisenberg commutation relations

X, P incompatible.

Recall generalized uncertainty relation

$$\Delta A\,\Delta B \geq \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|$$

$$\Rightarrow \Delta x\,\Delta p \geq \frac{\hbar}{2}$$

x, p in<u>compatible</u> $\Rightarrow$ eigenstates of x are not eigenstates of p.