Position observable X

Review of what has gone before:

observable $A \to$ operator $\hat{A}$

$A|a_n\rangle = a_n|a_n\rangle$ eigenstates = states of definite A

$\sum_n |a_n\rangle\langle a_n| = \mathbb{1}$ completeness

$|\psi\rangle = \sum |a_n\rangle\langle a_n|\psi\rangle =$

Define $\psi_n = \langle a_n|\psi\rangle$ = prob. amplitude

$|\psi\rangle = \sum \psi_n |a_n\rangle \quad \to$ indeterminacle w.r.t. A

$\langle\psi|\psi\rangle = \sum \langle\psi|a_n\rangle\langle a_n|\psi\rangle = \sum \psi_n^* \psi_n = \sum |\psi_n|^2 = 1$

$|\psi_n|^2$ = probability of measuring a_n

$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \sum_n \langle\psi|\underbrace{\hat{A}|a_n\rangle}_{a_n|a_n\rangle}\underbrace{\langle a_n|\psi\rangle}_{\psi_n} = \sum a_n\langle\psi|a_n\rangle\psi_n$

$= \sum |\psi_n|^2 a_n$ ✓

operator $\hat{X}$

$$\hat{X}|x\rangle = x|x\rangle$$

eigenvalue x = possible result of a measurement of position
= <u>any</u> real number

$|x\rangle$ = state of perfectly well-defined position (idealization)

Completeness [since <u>all</u> values of x are allowed, $\Sigma \to \int$]

$$\int dx\, |x\rangle\langle x| = \mathbb{1}$$

Then $|\psi\rangle = \int dx\, |x\rangle \langle x|\psi\rangle$

Define $\psi(x) = \langle x|\psi\rangle$ = a <u>number</u> that depends on ψ and x
ie a function of x!

$|\psi\rangle = \int dx\, \psi(x)\, |x\rangle$ = state of indeterminate position

$\psi(x)$ = probability amplitude, or wave function

Suppose $|\psi\rangle$ is normalized

$$1 = \langle\psi|\psi\rangle = \int dx \, \langle\psi|x\rangle\langle x|\psi\rangle$$

$$= \int dx \, \psi^*(x)\, \psi(x)$$

$$= \int dx \, |\psi(x)|^2$$

$|\psi(x)|^2$ = probability density of obtaining x

A fcn that obeys $\int dx \, |\psi(x)|^2$ is called "square integrable" or "normalizable"

$$\psi(x) \xrightarrow[x \to \pm\infty]{} 0 \quad \text{faster than} \quad \frac{1}{\sqrt{|x|}}$$

How does $\hat{x}$ act on a wavefunction $\psi(x)$?

let $|\psi\rangle = \int dx\, \psi(x) |x\rangle$

$$\hat{x}|\psi\rangle = \int dx\, \psi(x)\, \hat{x}|x\rangle = \int dx\, \psi(x)\, x\, |x\rangle$$

so if $|\psi\rangle \longrightarrow \psi(x)$ then $\hat{x}|\psi\rangle \longrightarrow x\,\psi(x)$

ie $\hat{x}$ multiplies $\psi(x)$ by x

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \int dx\, \underbrace{\langle\psi|\hat{x}|x\rangle}_{\underbrace{\langle\psi|x\rangle}_{\psi^*(x)}\, x}\, \underbrace{\langle x|\psi\rangle}_{\psi(x)}$$

$$= \int dx\, \psi^*(x)\, x\, \psi(x)$$

[as in 2140]

For self: $\hat{x} = \int dx\,dx'\, |x\rangle \underbrace{\langle x|\hat{x}|x'\rangle}_{x\,\delta(x-x')} \langle x'|$

$= \int dx\, |x\rangle\, x\, \langle x|$

ie diagonal

What about orthonormality of $|x\rangle$?

Define $\langle x'|x\rangle = \delta(x',x)$, a fn of x, x'

If $x \neq x'$, then orthogonal $\Rightarrow \delta(x',x) = 0$

what is $\delta(x,x) = ?$

Consider

$$\psi(x') = \langle x'|\psi\rangle = \langle x'| \int_{-\infty}^{\infty} dx\, \psi(x)\, |x\rangle = \int_{-\infty}^{\infty} dx\, \psi(x)\, \delta(x',x)$$

- integral vanishes if $x \neq x'$.
- nonzero on set of measure zero.
- $\delta(x',x)$ must be ∞ at $x = x'$.

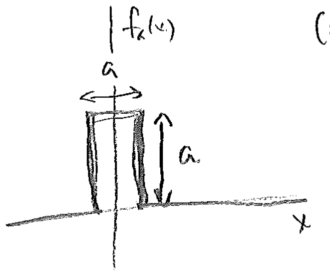
Define Dirac delta function $\delta(x)$ [really a distribution]

and that
$$\begin{cases} ① & \delta(x) = 0 \quad \text{if } x \neq 0 \\ ② & \int_{-\infty}^{\infty} \delta(x)\, dx = 1 \end{cases}$$

$\delta(x)$ is usually represented as a limit of a family of functions

e.g.

$$\delta(x) = \lim_{a \to 0} R_a(x), \qquad R_a(x) = \begin{cases} 0, & |x| > \tfrac{1}{2}a \\ \tfrac{1}{a}, & |x| < \tfrac{1}{2}a \end{cases}$$



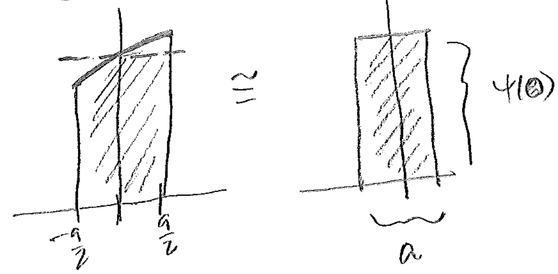
(rectangle)

① Let $x_0 \neq 0$, then $\lim_{a \to 0} R_a(x_0) = 0$ [choose $a < 2|x_0|$]

② $\int_{-\infty}^{\infty} R_a(x)\, dx = 1$ so $\int_{-\infty}^{\infty} \lim_{a \to 0} R_a(x)\, dx = 1$ ✓

Let $\psi(x)$ be an arbitrary function

Consider $\displaystyle\int_{-\infty}^{\infty} R_a(x)\,\psi(x)\,dx = \frac{1}{a}\underbrace{\int_{-a/2}^{a/2}\psi(x)\,dx}\ \cong\ \psi(0)$



$$\lim_{a\to 0}\int_{-\infty}^{\infty} R_a(x)\,\psi(x)\,dx = \psi(0)$$

$$\int_{-\infty}^{\infty} \delta(x)\,\psi(x)\,dx = \psi(0)$$

more generally

$$\int_{-\infty}^{\infty} \delta(x-x')\,\psi(x)\,dx = \psi(x')$$

[This is exactly what we need]

$$\delta(x',x) = \delta(x-x')$$

$\langle x'|x\rangle = \delta(x-x') = \delta(x'-x)$ [true too since $\langle x'|x\rangle = \langle x|x'\rangle$]

"Dirac delta fn normalization"

(NB $\hat{X}$ eigen-states not normalizable $\langle x|x\rangle \neq 1$)

~~2826~~ 26-8

Claim: position operator hermitian. How prove?

Hermitian conjugation: [or shd this be problem?]

Recall $\langle \psi | A^{+} | \phi \rangle = \langle \phi | A | \psi \rangle^{*}$

In position space representation

$$\int dx\, \psi^{*}(x) \, (\hat{A}^{+})_{pos.} \, \phi(x) = \left[\int dx\, \phi^{*}(x) \, (\hat{A})_{pos.} \, \psi(x) \right]^{*}$$

Here

$$\int dx\, \psi^{*}(x) \, (\hat{x}^{+})_{pos} \, \phi(x) = \left[\int dx\, \phi^{*}(x) \, x \, \psi(x) \right]^{*}$$

$$= \int dx\, \psi^{*}(x) \, x^{*} \, \phi(x)$$

$$\Rightarrow (\hat{x}^{+})_{pos} = x = (\hat{x})_{pos}$$

$$\boxed{\hat{x}^{+} = \hat{x}}$$

$\hat{X}$ eigenstates in position space

$$|x_0\rangle = \int dx \, \langle x|x_0\rangle \, |x\rangle = \int dx \, \delta(x-x_0) \, |x\rangle$$

not normalizable

$$\int dx \, |\psi|^2 = \int dx \, \delta(x-x_0) \, \delta(x-x_0) = \delta(0)$$

Alt,

$$\hat{X}|x_0\rangle = x_0|x_0\rangle$$

$$\Rightarrow \langle x|\hat{X}|x\rangle\langle x|x_0\rangle = x_0\langle x|x_0\rangle$$

ie $\quad x\langle x|x_0\rangle = x_0\langle x|x_0\rangle$

$$(x-x_0)\langle x|x_0\rangle = 0 \qquad \Rightarrow \qquad \langle x|x_0\rangle = \delta(x-x_0)$$

$$\hat{X} = \int dx\, dx' \, |x\rangle \underbrace{\langle x|\hat{X}|x'\rangle}_{x'\delta(x-x')}\langle x'| = \int dx \, |x\rangle \, x \langle x|$$

ie diagonal operator

$\langle\psi|\psi\rangle = 1 \Rightarrow |\psi\rangle$ unitless

$\int|\psi|^2 dx = 1 \Rightarrow \psi$ has units $\frac{1}{\sqrt{L}}$

$\int \delta(x-x')dx = 1 \Rightarrow \delta(x)$ has units $\frac{1}{L}$

$\langle x|x'\rangle = \delta(x-x') \Rightarrow |x\rangle$ has units $\frac{1}{\sqrt{L}}$

$\psi(x) = \langle x|\psi\rangle$ has units $\frac{1}{\sqrt{L}}$ ✓

$$|\psi\rangle = \underbrace{\int dx}_{L} \, \underbrace{\psi(x)}_{\frac{1}{\sqrt{L}}} \, \underbrace{|x\rangle}_{\frac{1}{\sqrt{L}}} \sim \text{unitless}$$

$|p\rangle$ has units $\frac{1}{\sqrt{\text{mom}}}$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

has units $\frac{1}{\sqrt{L\cdot\text{mom}}}$

$\hbar$ has units $L\cdot\text{mom}$

$$(\hat{A})_{pos} = \langle x | \hat{A} | x' \rangle = A_{xx'}$$

$$\hat{A} = \int dx\, dx' \; |x\rangle A_{xx'} \langle x'|$$

$$\hat{X}_{xx'} = x\,\delta(x-x')$$

$$\hat{P}_{xx'} = \langle x | \hat{P} | x' \rangle = \int dp\, dp' \langle x|p\rangle p\, \delta(p-p') \langle p|x'\rangle$$

$$= \frac{1}{2\pi\hbar} \int dp \; p \; e^{\frac{i(x-x')p}{\hbar}}$$

$$= \frac{\hbar}{i} \frac{d}{dx} \; \frac{1}{2\pi\hbar} \int dp \; e^{\frac{i(x-x')p}{\hbar}}$$

$$= \frac{\hbar}{i} \delta'(x-x')$$

$$\hat{p} = \int dx\, dx' \; |x\rangle \frac{\hbar}{i} \delta'(x-x') \langle x'|$$

$$\hat{A} = \sum |E_m\rangle A_{mn} \langle E_n|$$

$$A_{mn} = \langle E_m | \hat{A} | E_n \rangle$$

Save for exam

Eigenfn of $\hat{X}$ in momentum space:

$$\psi_{x_0}(x) = \delta(x - x_0)$$

$$\phi_{x_0}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \; \delta(x - x_0)\, e^{-\frac{ipx}{\hbar}} = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-\frac{ipx_0}{\hbar}}$$

(a)

$$\hat{X}\,\phi_{x_0}(p) = i\hbar \frac{d\phi}{dp} = p_0 \phi$$

$$\Rightarrow \phi(p) \sim e^{-\frac{ipx_0}{\hbar}}$$