

Precession of spin in a magnetic field

Classical analysis

$$E = -\vec{\mu} \cdot \vec{B}$$

$$\text{torque } \vec{\tau} = \vec{\mu} \times \vec{B}$$

In absence of external torques, angular momentum $\vec{J}$ is conserved

$$\frac{d\vec{J}}{dt} = \vec{\tau} \quad (\text{classically})$$

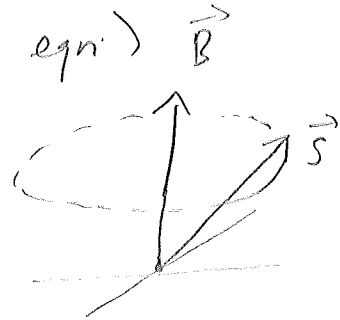
Consider a free spin $\vec{J} = \vec{S}$

$$\vec{\mu} = g \frac{q}{2m} \vec{S}$$

electron: $q = -e, g \approx 2$
proton: $q = +e, g \approx 5.6$

$$\frac{d\vec{S}}{dt} = \frac{gq}{2m} \vec{S} \times \vec{B} \quad (\text{gyroscopic eqn})$$

$$\text{Let } \vec{B} = B \hat{z}$$



$$\begin{cases} \frac{dS_z}{dt} = 0 \end{cases} \Rightarrow S_z = \text{const}$$

$$\begin{cases} \frac{dS_x}{dt} = \omega_L S_y \\ \frac{dS_y}{dt} = -\omega_L S_x \end{cases} \quad \text{where } \omega_L = \frac{gqB}{2m} = \text{Larmor frequency}$$

$$\Rightarrow \frac{d^2 S_x}{dt^2} + \omega_L^2 S_x = 0 \Rightarrow \begin{cases} S_x = S_x(0) \cos(\omega_L t) \\ S_y = -S_x(0) \sin(\omega_L t) \\ S_z = S_z(0) \end{cases}$$

$$\text{Larmor period } T = \frac{2\pi}{\omega_L}$$

clockwise rotation

quantum analysis

$$E = -\vec{\mu} \cdot \vec{B} = -\left(\frac{g\mu_B}{2m}\right) S_z = -\omega_L S_z$$

$$\Rightarrow \hat{H} = -\omega_L \hat{S}_z$$

H, S_z compatible $\Rightarrow$ share eigenstates

$$H|\pm\rangle_z = \mp \frac{\hbar\omega_L}{2} |\pm\rangle_z \quad \text{ie} \quad E_{\pm} = \mp \frac{\hbar\omega_L}{2}$$

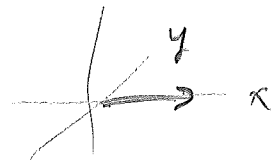
Initial state

$$|\psi(0)\rangle = \alpha|+\rangle_z + \beta|-\rangle_z$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle = \alpha e^{-\frac{iE_+t}{\hbar}} |+\rangle_z + \beta e^{-\frac{iE_-t}{\hbar}} |-\rangle_z$$

$$= \alpha e^{+i\frac{\omega_L t}{2}} |+\rangle_z + \beta e^{-i\frac{\omega_L t}{2}} |-\rangle_z$$

$$\text{Spin } \psi(t) = \begin{pmatrix} \alpha e^{i\frac{\omega_L t}{2}} \\ \beta e^{-i\frac{\omega_L t}{2}} \end{pmatrix} = e^{i\frac{\omega_L t}{2}} \begin{pmatrix} \alpha \\ \beta e^{-i\omega_L t} \end{pmatrix}$$



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Suppose initial state is $|+\rangle_x \Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}}$

$$\psi(t) = e^{i\omega_L \frac{t}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\omega_L t} \end{pmatrix}$$

After 1 period, $t = \frac{2\pi}{\omega_L} \Rightarrow e^{-i\pi} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim |+\rangle_x$

After $\frac{1}{2}$ period, $t = \frac{\pi}{\omega_L} \Rightarrow e^{-i\frac{\pi}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sim |-\rangle_x$

After $\frac{1}{4}$ period $t = \frac{\pi}{2\omega_L} \Rightarrow e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \sim |-\rangle_y$

clockwise
rotation

Problem: calc $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$

for $\psi(t) = e^{i\omega_L \frac{t}{2}} \begin{pmatrix} \alpha \\ \beta e^{-i\omega_L t} \end{pmatrix}$

verify Heisenberg eqns.