Time evolution of the state vector

Let the state of a system at $t=0$ be $|\psi\rangle$.
What is the state $|\psi(t)\rangle$ at time t?

Let $U(t)$ be a unitary operator that implements time translation (a symmetry)

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

"The Hamiltonian (energy) operator is the generator of time translation"

$$U(t) = e^{-\frac{iHt}{\hbar}} \quad \text{so} \quad |\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}\,|\psi(0)\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{iHt}{\hbar}\right)^n$$

$U(t)$ is unitary provided H is hermitian

Proof

$$U^\dagger(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{iHt}{\hbar}\right)^{n\,\dagger} = \sum \frac{1}{n!}\left(\frac{iHt}{\hbar}\right)^n = e^{\frac{iHt}{\hbar}}$$

$$U^\dagger U = e^{\frac{iHt}{\hbar}}\, e^{-\frac{iHt}{\hbar}} = \underline{1}$$

[Exercise: prove $e^{\alpha\hat{A}} e^{\beta\hat{A}} = e^{(\alpha+\beta)\hat{A}}$]

Unitarity of $U \Rightarrow$ total probability is conserved
(ie state vector, once normalized, remains normalized)

$$\langle \psi(t) | \psi(t) \rangle = \langle \psi(0) | U^\dagger U | \psi(0) \rangle = \langle \psi(0) | \psi(0) \rangle = 1$$

[otherwise prob of finding ptcl $\neq$ 100%]

Derive Schrodinger eqn:

$$\frac{d}{dt}|\psi(t)\rangle = \frac{dU}{dt}|\psi(0)\rangle$$

$$\frac{dU}{dt} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{iHt}{\hbar}\right)^n$$

$$= \sum_{\cancel{n=0} \to n=1}^{\infty}\frac{n}{n!}\left(-\frac{iH}{\hbar}\right)\left(-\frac{iHt}{\hbar}\right)^{n-1}$$

$$= -\frac{i}{\hbar}H\underbrace{\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(-\frac{iHt}{\hbar}\right)^{n-1}}_{\underbrace{\sum_{n'=0}^{\infty}\frac{1}{n'!}\left(-\frac{iHt}{\hbar}\right)^{n'}}_{U(t)}} = -\frac{i}{\hbar}HU$$

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}HU|\psi(0)\rangle$$

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi(t)\rangle$$ time dependent Schrodinger eqn

The Hamiltonian is also the energy operator

Energy eigenstates $H|E_n\rangle = E_n|E_n\rangle$

These are a complete set of states, ie a basis

Let initial state be decomposed in energy basis

$$|\psi(0)\rangle = \sum c_n |E_n\rangle$$

Then

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{iHt}{\hbar}\right)^m \sum_n c_n |E_n\rangle$$

$$= \sum_n c_n \sum_{m=0}^{\infty} \frac{1}{m!} \underbrace{\left(-\frac{iHt}{\hbar}\right)^m |E_n\rangle}_{\left(-\frac{iE_nt}{\hbar}\right)^m |E_n\rangle}$$

$$= \sum_n c_n e^{-\frac{iE_nt}{\hbar}} |E_n\rangle$$

so each energy eigenstate gets multiplied by a complex phase depending on the eigenvalue

If initial state is an eigenstate of $\hat{H}$

i.e. $|\psi(0)\rangle = |E_n\rangle$

then $|\psi(t)\rangle = e^{-\frac{iE_n t}{\hbar}} |E_n\rangle$ only overall phase changes

$\Rightarrow$ expectation value will not depend on time

$\langle\psi(t)|A|\psi(t)\rangle$ time $\Rightarrow$ "stationary state"

$= \langle E_n| e^{\frac{iE_n t}{\hbar}} A e^{-\frac{iE_n t}{\hbar}} |E_n\rangle = \langle E_n|A|E_n\rangle = \langle\psi(0)|A|\psi(0)\rangle$

If $|\psi(0)\rangle$ is not an eigenstate

there will be relative phases

that don't cancel out in expectation value

$\Rightarrow$ not stationary state

To show: $|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |\psi(0)\rangle$

$= \sum_0^\infty \frac{1}{n!} \left(-\frac{i\hat{H}t}{\hbar}\right)^n |\psi(0)\rangle$

$\frac{d}{dt}|\psi(t)\rangle = \sum_0^\infty \frac{n}{n!} \left(-\frac{i\hat{H}}{\hbar}\right)^{1} \left(-\frac{i\hat{H}t}{\hbar}\right)^{n-1} |\psi(0)\rangle$

$= -\frac{i\hat{H}}{\hbar} \sum_1^\infty \frac{1}{(n-1)!} \left(-\frac{i\hat{H}t}{\hbar}\right)^{n-1} |\psi(0)\rangle$

$\sum \frac{1}{m!} \left(-\frac{i\hat{H}t}{\hbar}\right)^m |\psi(0)\rangle$

$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H} |\psi(t)\rangle$ ✓

How do expectation values change in time?

14-6

$$\langle A(t)\rangle = \langle \psi(t)|\, \hat{A} \,|\psi(t)\rangle$$

state changes over time

operator does not (generally)

$$= \langle \psi(0)|\, \hat{U}^\dagger \hat{A}\, \hat{U} \,|\psi(0)\rangle$$

$$\frac{d}{dt}\langle A(t)\rangle = \langle \psi(0)|\, U^\dagger A \underbrace{\frac{dU}{dt}}_{-\frac{i}{\hbar}HU} |\psi(0)\rangle + \langle \psi(0)|\, \underbrace{\frac{d}{dt}U^\dagger}_{\frac{i}{\hbar}U^\dagger H} A\, U\,|\psi(0)\rangle$$

(found before)

(hermitian conj)

$$= -\frac{i}{\hbar}\langle\psi(0)|\, U^\dagger A H U\,|\psi(0)\rangle + \frac{i}{\hbar}\langle\psi(0)|\, U^\dagger H A\, U\,|\psi(0)\rangle$$

$$\boxed{\frac{d}{dt}\langle A\rangle = -\frac{i}{\hbar}\langle\psi(t)|\,[A,H]\,|\psi(t)\rangle}$$

Heisenberg eqn

If operator A has explicit time dependence then add term $\langle\psi(t)|\, \frac{\partial A}{\partial t}\,|\psi(t)\rangle$

If $[A,H]$ then $\frac{d}{dt}\langle A\rangle = 0$ i.e. $\langle A\rangle$ = <u>conserved</u>

Conserved quantities commute w/ Hamiltonian, i.e. are compatible w/ H

$[H,H]=0 \Rightarrow$ energy conservation is built into Schrodinger eqn

[prob 3.31 g 20]