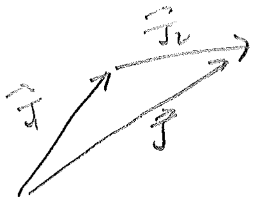
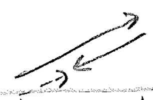
Addition of angular momentum

Classically, the angular momentum $\vec{J}_1$ of a particle is described by a vector.
The angular momentum of two particles is described by vector addition: $\vec{J} = \vec{J}_1 + \vec{J}_2$



$$\underbrace{|\vec{J}_1| - |\vec{J}_2|}_{\substack{\text{assume} \\ |\vec{J}_2| < |\vec{J}_1|}} \le |\vec{J}| \le |\vec{J}_1| + |\vec{J}_2|$$



~~Quantum mechanics: $J_1 = j_1\hbar$, $J_2 = j_2\hbar$, $J = j\hbar$ where $j = j_1+j_2, j_1+j_2-1, j_1+j_2-2, \ldots, |j_1-j_2|$~~
misleading because what is J_1? Does $(J_1)^2 = J^2 \ldots$

relevant ops $\quad J_1^2, J_{1z}, \quad J_2^2, J_{2z}, \quad J^2, J_z = J_{1z} + J_{2z}$

Quantum mechanically, the angular momentum of a particle is described by a state in a $(2j+1)$ dimensional space

$$\sum_{m=-j}^{j} a_m \underbrace{|j, m\rangle}_{\text{basis of space}}$$

The angular momentum of two particles is described by a state in a $(2j_1+1)\times(2j_2+1)$ dim'l <u>product space</u>

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} a_{m_1 m_2} \underbrace{|j_1, m_1; j_2, m_2\rangle}_{\text{basis of product space}}$$

This product space can be decomposed into a sum of spaces of the total angular momentum

$$(\text{spin } j_1) \otimes (\text{spin } j_2)$$

$$= \text{spin}(j_1+j_2) \oplus \text{spin}(j_1+j_2-1) \oplus \cdots \oplus \text{spin}|j_1-j_2|$$

Do counting: $\quad 2 \otimes 2 = 3 + 1$
$\quad 2 \otimes 3 = 4 + 2$

$\vec{J}_1, \vec{J}_2, \vec{J}$ are operators that act on these spaces

$\vec{J} = \vec{J}_1 + \vec{J}_2$ so $J_z = J_{1z} + J_{2z}$, $J_x = J_{1x} + J_{2x}$, $J_y = J_{1y} + J_{2y}$ $\Rightarrow$ $J_- = J_{1-} + J_{2-}$, $J_+ = J_{1+} + J_{2+}$

Relevant operators: $J_1^2, J_{1z}; J_2^2, J_{2z}; J^2, J_z$

(spin 1) $\otimes$ (spin 2) = (spin 3) $\oplus$ (spin 2) $\oplus$ (spin 1)

Example: $j_1 = 1$, $j_1 = 2$ so

(spin 1) space spanned by $|j_1, m_1\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle$

(spin 2) space spanned by $|j_2\, m_2\rangle = |2,2\rangle, |2,1\rangle, |2,0\rangle, |2,-1\rangle\, |2,-2\rangle$

product space is 15-dimensional ($\underset{\sim}{3} \otimes \underset{\sim}{5} = \underset{\sim}{7} \oplus \underset{\sim}{5} \oplus \underset{\sim}{3}$ ← notation; ↖ label reps by their multiplicity

eigenstates of $J_1^2, J_{1z}, J_2^2, J_{2z}$

$\Rightarrow$ 15 basis states, labelled by $|j_1 m_1; j_2 m_2\rangle$ or $|m_1; m_2\rangle$ for short

$|j, m\rangle$ ← eigenstates of J^2, J_z

$\lvert m_1; m_2\rangle$			$m = m_1 + m_2$	$\lvert j, m\rangle$		
$\lvert 1;2\rangle$			3	$\lvert 3,3\rangle$		
$\lvert 1;1\rangle$	$\lvert 0;2\rangle$		2	$\lvert 3,2\rangle$	$\lvert 2,2\rangle$	
$\lvert 1;0\rangle$	$\lvert 0;1\rangle$	$\lvert -1;2\rangle$	1	$\lvert 3,1\rangle$	$\lvert 2,1\rangle$	$\lvert 1,1\rangle$
$\lvert 1;-1\rangle$	$\lvert 0;0\rangle$	$\lvert -1;1\rangle$	0	$\lvert 3,0\rangle$	$\lvert 2,0\rangle$	$\lvert 1,0\rangle$
$\lvert 1;-2\rangle$	$\lvert 0;-1\rangle$	$\lvert -1;0\rangle$	-1	$\lvert 3,-1\rangle$	$\lvert 2,-1\rangle$	$\lvert 1,-1\rangle$
	$\lvert 0;-2\rangle$	$\lvert -1;-1\rangle$	-2	$\lvert 3,-2\rangle$	$\lvert 2,-2\rangle$	
		$\lvert -1;-2\rangle$	-3	$\lvert 3,-3\rangle$		

m_2 m_1

The states in any row on the right are linear combinations of states in the same row on the left (and vice versa)

$$|j, m\rangle = \sum_{m_1, m_2} C^{j_1 j_2 j}_{m_1 m_2 m} \, |m_1; m_2\rangle$$

↑ Clebsch-Gordan coefficients

eg $|3,1\rangle = C^{123}_{101}\, |1;0\rangle + C^{123}_{011}\, |0;1\rangle + C^{123}_{-1,2,1}\, |-1;2\rangle$

(e.g. 2 electron spins)
electron + proton

Simpler example $j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$

$$(\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) = (\text{spin } 1) \oplus (\text{spin } 0)$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$$

$\lvert m_1 ; m_2\rangle$	$m = m_1 + m_2$	$\lvert j, m\rangle$
$\lvert \frac{1}{2} ; \frac{1}{2}\rangle$	$=$	$\lvert 1, 1\rangle$
$\lvert +\frac{1}{2} ; -\frac{1}{2}\rangle$ $\lvert -\frac{1}{2} ; +\frac{1}{2}\rangle$	$\xrightarrow[\text{combinations}]{\text{linear}}$	$\lvert 1, 0\rangle$ $\lvert 0, 0\rangle$
$\lvert -\frac{1}{2} ; -\frac{1}{2}\rangle$	$=$	$\lvert 1, -1\rangle$

These are eigenstates of $J_1^2, J_{1z}, J_2^2, J_{2z}$

These are also eigenstates of J_z because $J_z = J_{1z} + J_{2z}$

But they are not (necessarily) eigenstates of J^2
(because $[J^2, J_{1z}] \neq 0$
$[J^2, J_{2z}] \neq 0$)

These are eigenstates of J^2 and J_z

They are not (necessarily) eigenstates of J_{1z} and J_{2z}
(because $[J^2, J_{1z}] \neq 0$
$[J^2, J_{2z}] \neq 0$)

Clearly $|1,1\rangle = |\tfrac{1}{2};\tfrac{1}{2}\rangle$

and $|1,-1\rangle = |-\tfrac{1}{2};-\tfrac{1}{2}\rangle$

but what linear combinations comprise $|1,0\rangle$ and $|0,0\rangle$?

Recall $J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$ (ignore $\hbar$)

so $J_-|1,1\rangle = \sqrt{2}\,|1,0\rangle$

Also $J_-|\tfrac{1}{2};\tfrac{1}{2}\rangle = (\underset{\substack{\uparrow \\ \text{acts on 1st label}}}{J_{1-}} + \underset{\substack{\uparrow \\ \text{acts on 2nd}}}{J_{2-}})\,|\tfrac{1}{2};\tfrac{1}{2}\rangle$

$= |-\tfrac{1}{2};\tfrac{1}{2}\rangle + |\tfrac{1}{2};-\tfrac{1}{2}\rangle$

$\therefore \quad |1,0\rangle = \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle$ (NB. automatically normalized)

CG coeffs

$\longrightarrow$ see chart.

Triplet is
$$|1,1\rangle = |\tfrac{1}{2};\tfrac{1}{2}\rangle$$
$$|1,0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2};-\tfrac{1}{2}\rangle + |-\tfrac{1}{2};\tfrac{1}{2}\rangle\right)$$
$$|1,-1\rangle = |-\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

all symmetric under $m_1 \Leftrightarrow m_2$

$\downarrow$

[hence cannot describe gnd st of helium
but can describe excited state $\Rightarrow$ orthohelium]
$\hookrightarrow$ describes p + e OR

What about $|0;0\rangle$? We know $J_+|0;0\rangle = 0$

Consider $|\psi\rangle = \alpha|\tfrac{1}{2};-\tfrac{1}{2}\rangle + \beta|-\tfrac{1}{2};\tfrac{1}{2}\rangle$

$$J_+|\psi\rangle = (J_{1+} + J_{2+})|\psi\rangle$$

$$= \alpha|\tfrac{1}{2};\tfrac{1}{2}\rangle + \beta|\tfrac{1}{2};\tfrac{1}{2}\rangle = 0 \Rightarrow \beta = -\alpha$$

Normalize $\Rightarrow \alpha = \frac{1}{\sqrt{2}}$

singlet $$|0;0\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{|-\tfrac{1}{2};\tfrac{1}{2}\rangle - |\tfrac{1}{2};-\tfrac{1}{2}\rangle}_{\text{antisymmetric under } m_1 \leftrightarrow m_2}\Big)$$

[$\Rightarrow$ ground state of helium or excited state (parahelium)]

N.B. $|0;0\rangle$ is orthogonal to $|1;0\rangle$. Could have used this to determine α & β.

Prob: find CG coeffs for $\vec{3} \otimes \vec{2} = \vec{4} \oplus \vec{2}$

$$J^2 = (\vec{J}_1 + \vec{J}_2)\cdot(\vec{J}_1 + \vec{J}_2) = J_1^2 + J_2^2 + 2\vec{J}_1\cdot\vec{J}_2$$

$$= J_1^2 + J_2^2 + 2\big(J_{1z}J_{2z} + \underbrace{J_{1x}J_{2x}}_{\left(\frac{J_{1+}+J_{1-}}{2}\right)\left(\frac{J_{2+}+J_{2-}}{2}\right)} + \underbrace{J_{1y}J_{2y}}_{\left(\frac{J_{1+}-J_{1-}}{2i}\right)\left(\frac{J_{2+}-J_{2-}}{2i}\right)} \big)$$

$$= J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$ (did up to here in 2013)

Apply this to the triplet states

$$J^2\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle = \left(J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}\right)\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

$$= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) + 0 + 0\right)\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

$$= 2\hbar^2\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

so indeed $\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle = |1,0\rangle$

$$J^2\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle = \Big(\underbrace{\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)}_{\hbar^2}\Big)\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + 0 + \hbar^2\left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

$$J^2\left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle = \hbar^2\left|-\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + \hbar^2\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + 0$$

so neither $\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle$ nor $\left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle$ are eigenstates of J^2

$$J^2\left(\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle\right) = 2\hbar^2\left(\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle\right) \Rightarrow |1,0\rangle$$

$$J^2\left(\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle - \left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle\right) = 0 \Rightarrow |0,0\rangle$$

$\boxed{\text{Problem}}$ Calc $|\tfrac{3}{2}, m\rangle$, $|\tfrac{1}{2}, m\rangle$ in $(\text{spin } 1) \otimes (\text{spin } \tfrac{1}{2})$

$$J_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$$
$$J_+ |j, m\rangle = \sqrt{(j-m)(j+m+1)}\, |j, m+1\rangle$$

$$|\tfrac{3}{2}; \tfrac{3}{2}\rangle = |1; \tfrac{1}{2}\rangle$$
$$\sqrt{3}\, |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{2}\, |0; \tfrac{1}{2}\rangle + |1; -\tfrac{1}{2}\rangle$$
$$2\sqrt{3}\, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle = 2\, |-1; \tfrac{1}{2}\rangle + \sqrt{2}\, |0; -\tfrac{1}{2}\rangle + \sqrt{2}\, |0; -\tfrac{1}{2}\rangle$$
$$6\, |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = 2\, |-1, -\tfrac{1}{2}\rangle + 2\, |-1; -\tfrac{1}{2}\rangle + 2\, |-1; -\tfrac{1}{2}\rangle$$

$$\begin{cases} |\tfrac{3}{2}, \tfrac{3}{2}\rangle = |1; \tfrac{1}{2}\rangle \\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |0; \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\, |1; -\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |0; -\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\, |-1, \tfrac{1}{2}\rangle \\ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |-1; -\tfrac{1}{2}\rangle \end{cases}$$

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \alpha\, |0; +\tfrac{1}{2}\rangle + \beta\, |1; -\tfrac{1}{2}\rangle$$
$$J_+ |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \alpha\sqrt{2}\, |1; +\tfrac{1}{2}\rangle + \beta\, |1; \tfrac{1}{2}\rangle = 0 \implies \beta = -\sqrt{2}\,\alpha$$

$$\Rightarrow |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |0; \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |1; -\tfrac{1}{2}\rangle$$
$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |-1; \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |0; -\tfrac{1}{2}\rangle - \tfrac{2}{\sqrt{3}}\, |0; -\tfrac{1}{2}\rangle$$
$$= \sqrt{\tfrac{2}{3}}\, |-1; +\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}\, |0; -\tfrac{1}{2}\rangle$$

$$\begin{cases} |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |0; +\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |1; -\tfrac{1}{2}\rangle \\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\, |0; -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |-1; \tfrac{1}{2}\rangle \end{cases}$$

$J_-|1,1\rangle = \sqrt{2}|1,0\rangle$
$J_-|1,0\rangle = \sqrt{2}|1,-1\rangle$

$$J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

$$J^2\,|1;\tfrac{1}{2}\rangle = 2 + \tfrac{3}{4} + 2(1)(\tfrac{1}{2}) = \tfrac{15}{4} = \tfrac{3}{2}(\tfrac{3}{2}+1) \checkmark$$

$$J^2|\tfrac{3}{2},\tfrac{1}{2}\rangle = \left(\sqrt{\tfrac{2}{3}}|0;\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}|1;-\tfrac{1}{2}\rangle\right)$$

$$= (2+\tfrac{3}{4})|\tfrac{3}{2},\tfrac{1}{2}\rangle + \underbrace{\underbrace{\sqrt{\tfrac{2}{3}}\left(\sqrt{2}|1;-\tfrac{1}{2}\rangle\right) + \tfrac{1}{\sqrt{3}}\left(2(1)(-\tfrac{1}{2})|1;-\tfrac{1}{2}\rangle + \sqrt{2}|0;\tfrac{1}{2}\rangle\right)}_{\frac{1}{\sqrt{3}}|1;-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|0;\frac{1}{2}\rangle}}_{|\frac{3}{2},\frac{1}{2}\rangle}$$

$$= \tfrac{15}{4}|\tfrac{3}{2},\tfrac{1}{2}\rangle \checkmark$$

$$J^2|\tfrac{1}{2},\tfrac{1}{2}\rangle = J^2\left(\sqrt{\tfrac{1}{3}}|0;\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|1;-\tfrac{1}{2}\rangle\right)$$

$$= (2+\tfrac{3}{4})|\tfrac{1}{2},\tfrac{1}{2}\rangle + \underbrace{\underbrace{\sqrt{\tfrac{1}{3}}\left(\sqrt{2}|1;-\tfrac{1}{2}\rangle\right) - \sqrt{\tfrac{2}{3}}\left(2(1)(-\tfrac{1}{2})|1;-\tfrac{1}{2}\rangle + \sqrt{2}|0;\tfrac{1}{2}\rangle\right)}_{2\sqrt{\frac{2}{3}}|1;-\frac{1}{2}\rangle - 2\frac{1}{\sqrt{3}}|0;\frac{1}{2}\rangle}}_{-2|\frac{1}{2},\frac{1}{2}\rangle}$$

$$= \tfrac{3}{4}|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{2}(1+\tfrac{1}{2})|\tfrac{1}{2},\tfrac{1}{2}\rangle \checkmark$$