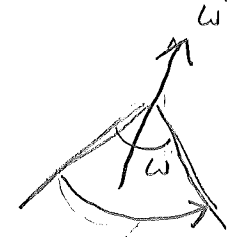
(eg rotation, translation in space or time)

A symmetry operation is implemented on Hilbert space by action of a unitary operator U



Consider a rotation $\vec{\omega}$ where $\begin{cases} \hat{\omega} = \text{axis of rotation} \\ |\vec{\omega}| = \text{angle of rotation} \end{cases}$

$$U = e^{-\frac{i\vec{\omega}\cdot\vec{J}}{\hbar}} \qquad \vec{J} = \text{angular momentum}$$

"Angular momentum operator is the generator of rotations"

Acting on spin-$\frac{1}{2}$ Hilbert space, $\vec{J} = \frac{\hbar}{2}\vec{\sigma}$

$$U = e^{-\frac{i\vec{\omega}\cdot\vec{\sigma}}{2}} = e^{-i\frac{\omega}{2}(\hat{\omega}\cdot\vec{\sigma})}$$

$$= \mathbb{1} - i\frac{\omega}{2}(\hat{\omega}\cdot\vec{\sigma}) - \frac{1}{2}\left(\frac{\omega}{2}\right)^2(\hat{\omega}\cdot\vec{\sigma})^2 + \frac{i}{3!}\left(\frac{\omega}{2}\right)^3(\hat{\omega}\cdot\vec{\sigma})^2 + \frac{1}{4!}\left(\frac{\omega}{2}\right)^4(\hat{\omega}\cdot\vec{\sigma})^4$$

Now $(\hat{\omega}\cdot\vec{\sigma})^2 = \mathbb{1}$ [exercise: show this]

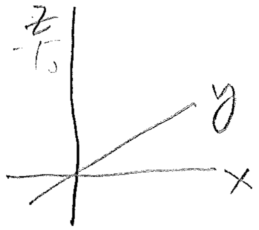
$$\begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U = \left(1 - \frac{1}{2}\left(\frac{\omega}{2}\right)^2 + \frac{1}{4!}\left(\frac{\omega}{2}\right)^4 + \dots\right)\mathbb{1} - i\left(\frac{\omega}{2} - \frac{1}{3!}\left(\frac{\omega}{2}\right)^2 + \dots\right)\hat{\omega}\vec{\sigma}$$

$$\boxed{U = \cos\frac{\omega}{2}\,\mathbb{1} - i\sin\frac{\omega}{2}(\hat{\omega}\cdot\vec{\sigma})}$$

Note $U^\dagger U = \left(\cos\frac{\omega}{2}\mathbb{1} + i\sin\frac{\omega}{2}(\hat{\omega}\cdot\vec{\sigma})\right)\left(\cos\frac{\omega}{2}\mathbb{1} - i\sin\frac{\omega}{2}(\hat{\omega}\cdot\vec{\sigma})\right)$

$$= \left(\cos^2\frac{\omega}{2} + \sin^2\frac{\omega}{2}\right)\mathbb{1} = \mathbb{1} \checkmark$$



$\hat{z}$-axis $\longrightarrow$ $\hat{x}$-axis

Rotation about $\hat{y}$ by $\frac{\pi}{2}$ $\Rightarrow$ $\vec{\omega} = \frac{\pi}{2}\hat{y}$

S_z-basis $\longrightarrow$ S_x-basis

$$U|+\rangle_z = |+\rangle_x$$

$$U|-\rangle_z = |-\rangle_z$$

$$U = e^{-\frac{i}{2}\left(\frac{\pi}{2}\hat{y}\right)\cdot\vec{\sigma}} = \left(\cos\frac{\pi}{4}\right)\mathbb{1} - i\left(\sin\frac{\pi}{4}\right)\sigma_y$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$|+\rangle_x = U|+\rangle_z = |+\rangle_z\,{}_z\langle +|U|+\rangle_z + |-\rangle_z\,{}_z\langle -|U|+\rangle_z$$

$$= \frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$|-\rangle_x = U|+\rangle_z = -\frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}$$

overall minus sign ambiguity

of rotation by 2π about any axis

$U = (\cos\pi)\,\mathbb{1} = -\mathbb{1}$ so $U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-\alpha\\ -\beta\end{pmatrix}$

Rotations do not commute in general

$U_1U_2 \neq U_2U_1$ [problem?]

Unitary operations preserve inner products

e.g. $|\psi'\rangle = U|\psi\rangle$

$|\phi'\rangle = U|\phi\rangle$

Then $\langle\phi'|\psi'\rangle = \langle\phi|U^\dagger U|\psi\rangle = \langle\phi|\psi\rangle$

Thus if S_z basis is orthonormal so is any basis obtained by rotating the S_z-basis

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~~Rotation thru 2π~~

$$\cancel{U = (\cos\pi)\,\mathbb{1} = -\mathbb{1} \quad \text{so} \quad U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}\alpha\\ \beta\end{pmatrix}}$$
