Matrix representations

Given an orthonormal basis, (e.g. the d eigenstates $|a_n\rangle$ of a Hermitian operator A)

any operator B can be represented as a $d \times d$ matrix

whose matrix elements are

$$B_{mn} = \langle a_m | B | a_n \rangle$$

The product of 2 operators is represented as the matrix product

$$\langle a_m | BC | a_n \rangle = \langle a_m | B \left(\sum_l |a_l\rangle\langle a_l| \right) C | a_n \rangle$$

$$= \sum_l \langle a_m | B | a_l \rangle \langle a_l | C | a_n \rangle$$

$$= \sum_l B_{ml} C_{ln}$$

Ket $|\psi\rangle$ represented by d-dim'l column vector w/ entries $\langle a_m|\psi\rangle = \psi_m$

Basis kets $|a_n\rangle \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow n\text{th position}$

$$|\psi\rangle = \sum_{n=1}^{d} \psi_n |a_n\rangle \rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix}$$

Bra $\langle\psi| \rightarrow$ d-dim'l row vector w/ entries $\langle\psi|a_m\rangle$

$$\langle a_n| \rightarrow (0 \cdots \underset{\uparrow n\text{th}}{1} \cdots 0)$$

$$\langle\psi| = \sum \langle a_n|\psi_n^* \rightarrow (\psi_1^* \cdots \psi_d^*)$$

NB $\langle\psi|$ obtained from $|\psi\rangle$ by complex conjugate transpose

Inner product

[alt: $\langle\psi|\phi\rangle = \sum\langle\psi|a_m\rangle\langle a_m|\phi\rangle$
$= \sum \psi_m^* \phi_m$]

$$\langle\psi|\phi\rangle = (\psi_1^* \cdots \psi_d^*)\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} = \sum_{n=1}^{d} \psi_n^* \phi_n$$

Symmetric: $\langle\phi|\psi\rangle = \sum \phi_n^* \psi_n = \left[\sum \psi_n \phi_n^*\right]^* = \langle\psi|\phi\rangle^*$

Nonnegative $\langle\psi|\psi\rangle = \sum \psi_n^* \psi_n = \sum |\psi_n|^2 \geq 0$

$\langle\psi|\psi\rangle = 0$ iff $\psi_n = 0 \Rightarrow |\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Basis is orthonormal

$$\langle a_m | a_n \rangle = (0 \cdots \underset{\substack{\uparrow \\ m\text{th}}}{1} \cdots 0) \begin{pmatrix} 0 \\ \vdots \\ 1 \leftarrow n\text{th} \\ \vdots \\ 0 \end{pmatrix} = \delta_{mn}$$

Complete

$$\sum_{n=1}^{d} |a_n\rangle\langle a_n| = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}(1\ 0 \cdots) + \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}(0\ 1 \cdots) + \cdots$$

$$= \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 1 & \\ & & \ddots \end{pmatrix} + \cdots = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} = \mathbb{1}$$

The operator A whose eigenstates form the basis is <u>diagonal</u> w/ eigen<u>values</u> as the entries

$$A_{mn} = \langle a_m | A | a_n \rangle = a_n \langle a_m | a_n \rangle = a_n \delta_{mn}$$

$$A \rightarrow \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{pmatrix}$$

Hermitian conjugate of an operator → transpose complex conj of matrix

$$|\chi\rangle = B|\psi\rangle \Rightarrow \langle\chi| = \langle\psi|B^\dagger$$

$$\langle a_n|\chi\rangle = \sum_m \langle a_n|B|a_m\rangle\langle a_m|\psi\rangle \qquad \langle\chi|a_n\rangle = \sum_m \langle\psi|a_m\rangle\langle a_m|B^\dagger|a_n\rangle$$

$$\chi_n = \sum_m B_{nm}\psi_m \qquad \chi_n^* = \sum_m \psi_m^* (B^\dagger)_{mn}$$

[matrix times col. vector]

$$\chi_n^* = \sum B_{nm}^* \psi_m^*$$
$$= \sum \psi_m^* (B^{*T})_{mn}$$

so $$(B^\dagger)_{mn} = (B^{*T})_{mn}$$

Spin-$\frac{1}{2}$ system: matrix representation in the S_z basis

$$|+\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \longrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$ ← called a "spinor"

$$B \longrightarrow \begin{pmatrix} \langle +|B|+\rangle & \langle +|B|-\rangle \\ \langle -|B|+\rangle & \langle -|B|-\rangle \end{pmatrix}$$

$$S_z \longrightarrow \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$ [diagonal as expected w/ eigenvalues]

$$S_x \longrightarrow \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

$$S_y \longrightarrow \begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}$$

[exercise: show $[S_x, S_y] = i\hbar S_z$
also $S^2 = S_x^2 + S_y^2 + S_z^2$
as mtx eqns]

$$S^2 \longrightarrow \begin{pmatrix} \frac{3\hbar^2}{4} & 0 \\ 0 & 3\hbar^2/4 \end{pmatrix}$$

Define $S_i = \frac{\hbar}{2}\sigma_i$ $\qquad \sigma_i$ = Pauli spin matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Observe $\sigma_i^\dagger = (\sigma_i^T)^* = \sigma_i$ Hermitian matrices
(because S_i is an observable)

~~[illegible]~~ exercise

ⓐ $[S_x, S_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2i & \\ & 2i \end{pmatrix}\right] = i\hbar S_z$

$[S_y, S_z] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} & 2i \\ 2i & \end{pmatrix}\right] = i\hbar S_x$

$[S_z, S_x] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right] = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\hbar S_y$

ⓑ $\frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \frac{3\hbar^2}{4}\mathbb{I}$ ✓

operator eigenvalue eqn

$$\mathcal{B}|\psi\rangle = \lambda|\psi\rangle$$

$$\sum_n \langle a_m|\mathcal{B}|a_n\rangle\langle a_n|\psi\rangle = \lambda\langle a_m|\psi\rangle$$

$$\sum_n \mathcal{B}_{mn}\psi_n = \lambda\psi_m$$

matrix eigenvalue eqn.

$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ has eigenvectors $|+\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$ and $|-\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$

w/ eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

$$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} \checkmark$$

$|\pm\rangle_z$ are <u>also</u> eigenstates of $S^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, both eigenval $\frac{3\hbar^2}{4}$

$\Rightarrow S^2$ and S_z are compatible $([S^2, S_z] = 0)$

What are the eigenvalues & eigenvectors of S_x?

[eigenvalues $= \pm \frac{\hbar}{2}$ because just a rotation of S_z]

Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ denote an eigenstate of S_x w/ eigenvalue λ

$$\frac{\hbar}{2}\sigma_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left(\frac{\hbar}{2}\sigma_x - \lambda \mathbb{1}\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

This eqn has trivial solution $\alpha = \beta = 0$.
This solution is unique unless determinant of matrix vanishes

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \implies \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \implies \lambda = \pm \frac{\hbar}{2}$$

Find eigenvectors $|+\rangle_x$ and $|-\rangle_x$

$\lambda = \frac{\hbar}{2} \Rightarrow \quad \frac{\hbar}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\alpha+\beta \\ \alpha-\beta \end{pmatrix} = 0 \quad \Rightarrow \alpha = \beta$

$|+\rangle_x = \begin{pmatrix} \beta \\ \beta \end{pmatrix}$

Normalize: ${}_x\langle +|+\rangle_x = 2|\alpha|^2 = 1 \quad \Rightarrow \quad \alpha = \frac{1}{\sqrt{2}} e^{i\theta}$

choose $\theta = 0 \quad \Rightarrow \quad |+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ spin-up in x-direction

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$\lambda = -\frac{\hbar}{2} \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \alpha+\beta \\ \alpha+\beta \end{pmatrix} = 0 \quad \Rightarrow \alpha = -\beta$

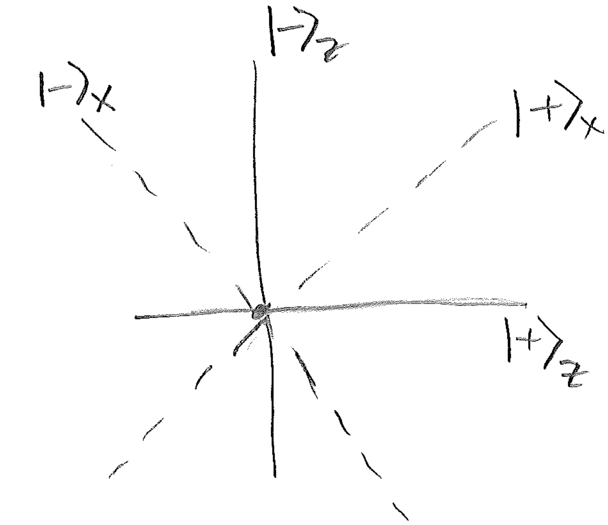
$|-\rangle_x = \begin{pmatrix} -\beta \\ \beta \end{pmatrix}$ normalize $\beta = \frac{1}{\sqrt{2}} e^{i\theta}$ choose $\theta = 0$ $\quad |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ +1 \end{pmatrix}$ [do this for a nice picture on next page]

Eigenstates $|\pm\rangle_x$ of S_x are distinct from those $|\pm\rangle_z$ of S_z because S_x + S_z are incompatible ($[S_x, S_z] \neq 0$)

Eigenstates $|\pm\rangle_x$ of S_x are also eigenstates of S^2 (both w/ eigenvalue $\frac{3\hbar^2}{4}$)
$\Rightarrow S^2$ and S_x are compatible : $[S^2, S_x] = 0$

$|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z$ thus ${}_x\langle \pm|S_z|\pm\rangle_x = 0$

$|-\rangle_x = -\frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z$ (earlier we saw ${}_z\langle \pm|S_x|\pm\rangle_z = 0$)



(In physical space:
spin up/down in S_z $\uparrow\downarrow$
spin up/down in S_x $\rightleftarrows$)

In Hilbert space

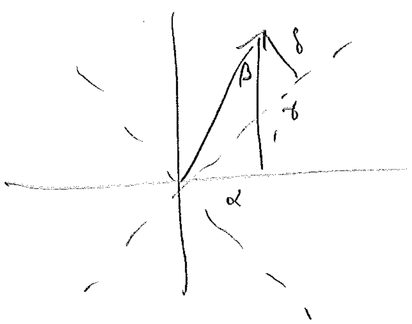
Note $|\pm\rangle_x$ forms an orthonormal basis (because $S_x^\dagger = S_x$)

Already normalized. ${}_x\langle +|-\rangle_x = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ +1 \end{pmatrix} = 0$

Any state can be written $|\psi\rangle = \gamma|+\rangle_x + \delta|-\rangle_x$

as well as $|\psi\rangle = \alpha|+\rangle_z + \beta|-\rangle_z$

each are equally good (democracy)



How are $\alpha, \beta, \gamma, \delta$ related?

$$\alpha = {}_z\langle +|\psi\rangle = \gamma\, {}_z\langle +|+\rangle_x + \delta\, {}_z\langle +|-\rangle_x$$

$$\beta = \gamma\, {}_z\langle -|+\rangle_x + \delta\, {}_z\langle -|-\rangle_x$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} {}_z\langle +|+\rangle_x & {}_z\langle +|-\rangle_x \\ {}_z\langle -|+\rangle_x & {}_z\langle -|-\rangle_x \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{U}\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

The matrix U which transforms components in one orthonormal basis to the components of another is <u>unitary</u>

A unitary matrix satisfies $U^{\dagger} = U^{-1}$

ie $U^{\dagger} U = \mathbb{1}$ [check]

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = U \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = U^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = U^{\dagger} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

If measure S_x on a state $|\psi\rangle$

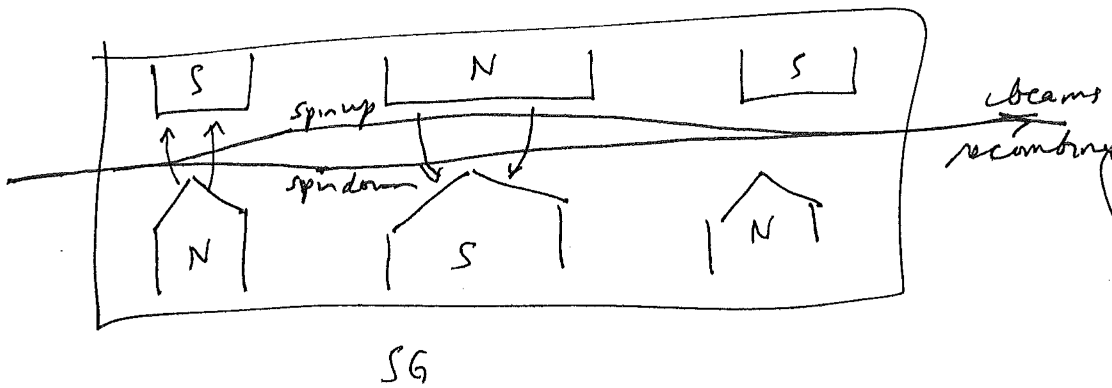
$$\text{prob}(S_x = \tfrac{\hbar}{2}) = |{}_x\langle + | \psi\rangle|^2 = |\gamma|^2 = \tfrac{1}{2}|\alpha + \beta|^2$$

$$\text{prob}(S_x = -\tfrac{\hbar}{2}) = |{}_x\langle - | \psi\rangle|^2 = |\delta|^2 = \tfrac{1}{2}|\alpha - \beta|^2$$

After measurement state is either $|+\rangle_x$ or $|-\rangle_x$

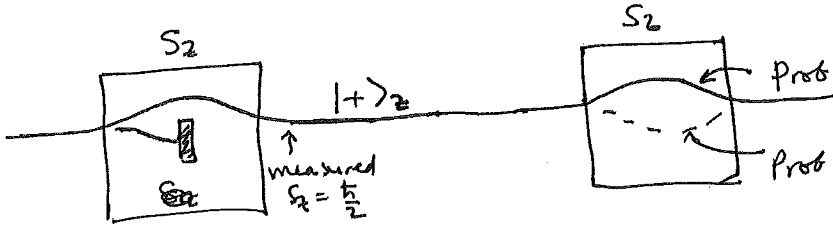
~~Repeats~~

8 ± 11



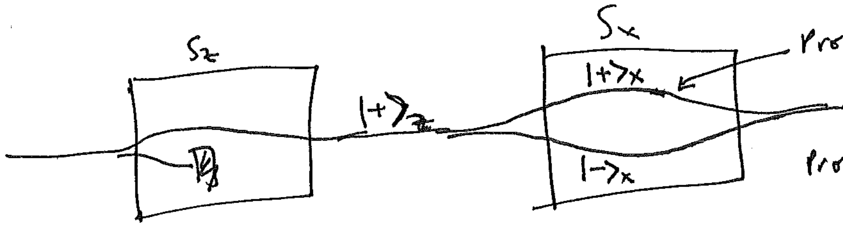
Can't tell which path took!

Repeated measurements



$\mathrm{Prob}(S_z = \frac{\hbar}{2}) = |{}_z\langle + | + \rangle_z|^2 = 1$

$\mathrm{Prob}(S_z = -\frac{\hbar}{2}) = |{}_z\langle - | + \rangle_z|^2 = 0$

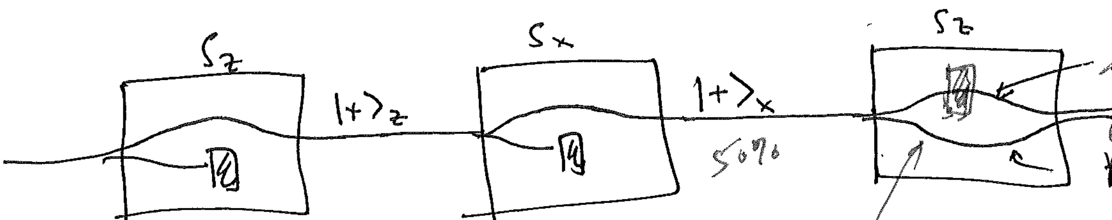


$\mathrm{Prob}(S_x = \frac{\hbar}{2}) = |{}_x\langle + | + \rangle_z|^2 = \frac{1}{2}$

$\mathrm{Prob}(S_x = -\frac{\hbar}{2}) = |{}_x\langle - | + \rangle_z|^2 = \frac{1}{2}$

${}_x\langle + | + \rangle_z = \left(\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$

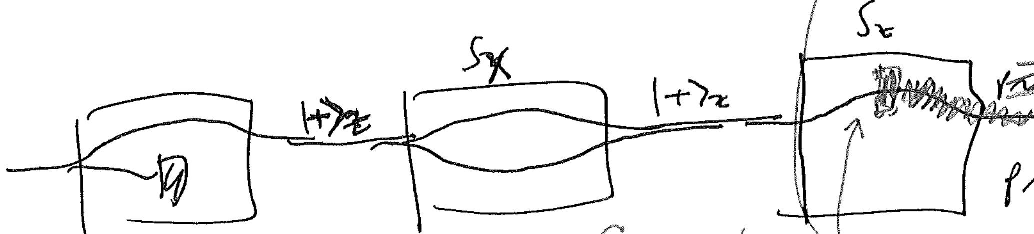
${}_x\langle - | + \rangle_z = \left(\frac{1}{\sqrt{2}} \ -\frac{1}{\sqrt{2}}\right)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}$



$\mathrm{Prob}(S_z = \frac{\hbar}{2}) = |{}_z\langle + | + \rangle_x|^2 = \frac{1}{2}$

~~50%~~ → 25%

$\mathrm{Prob}(S_z = -\frac{\hbar}{2}) = |{}_z\langle - | + \rangle_x|^2 = \frac{1}{2}$



~~$\mathrm{Prob}(S_z = \frac{\hbar}{2}) = 1$~~

$\mathrm{Prob}(S_z = -\frac{\hbar}{2}) = 0$

Long discussion of Copenhagen interpretation

analogous to crossed polarizers w/ 45° in two