$$\hat{S}_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$$

$|+\rangle = \uparrow$

$|-\rangle = \downarrow$

What about $\hat{S}_x$ and $\hat{S}_y$?

$|\pm\rangle$ are not eigenstates of $\hat{S}_x$ and $\hat{S}_y$

$\hat{S}_x$ and $\hat{S}_y$ are "incompatible" of $\hat{S}_z$.

Compatibility of observables

Two observables A + B are compatible if
$\exists$ a complete set of states that
have well-defined values for both A + B,
ie are simultaneous eigenstates of $\hat{A}$ and $\hat{B}$.
Then $\hat{A}$ and $\hat{B}$ commute.

Proof: Let $|\psi_n\rangle$ be a complete set of eigenstates: $\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$, $\hat{B}|\psi_n\rangle = b_n|\psi_n\rangle$

Consider an arbitrary state $|\psi\rangle = \sum \psi_n |\psi_n\rangle$

$\hat{A}\hat{B}|\psi\rangle = \sum \psi_n \hat{A}\hat{B}|\psi_n\rangle = \sum \psi_n \hat{A} b_n |\psi_n\rangle = \sum \psi_n b_n a_n |\psi_n\rangle$

$\hat{B}\hat{A}|\psi\rangle = \sum \psi_n a_n b_n |\psi_n\rangle$

Since $\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$ for an arbitrary state, $\boxed{\hat{A}\hat{B} = \hat{B}\hat{A}}$

QED

Commutation

Define $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

[HW on commutators]

A, B compatible $\Rightarrow [\hat{A}, \hat{B}] = 0$

If $[\hat{A}, \hat{B}] \neq 0 \Rightarrow$ A, B are incompatible

If $[\hat{A},\hat{B}] \neq 0$ then A & B are not compatible

$[\hat{A},\hat{B}] \sim \hbar$ (something)

because if $\hbar \to 0$, then quantum mechanics $\to$ classical
and all observables are classically compatible

If A & B are compatible then there exist
<u>some</u> states for which $\Delta A = 0$ and $\Delta B = 0$.
ie $\Delta A \, \Delta B = 0$

Recall from 2140, $\Delta x \, \Delta p \geq \frac{\hbar}{2}$ so x, p incompatible
Heisenberg uncertainty

Generalized uncertainty relations:

$$\Delta A \; \Delta B \geq \frac{1}{2} \left| \langle \psi | [\hat{A},\hat{B}] | \psi \rangle \right|$$

[HW]

From now on: drop hats from operators

Since S_x, S_y, S_z are incompatible w/ one another, their commutators must be nonzero.

$$[S_x, S_y] = \hbar\,(?)$$

[Guess $? = S_z$.
Not quite because S_z is herm while $[\,,\,]$ is antiherm]

$$[A,B]^\dagger = (AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB = [B,A] = -[A,B]$$

$$[S_x, S_y] = i\hbar S_z$$

[proof later for L.
prove using symmetry properties under rotations.]
cf Weinberg Lectures on QM

Rotate axes

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

"Commutation relations for angular momentum"

[Let's accept these as our axioms & derive consequences]

Using commutation relations, we will now derive the action of S_x and S_y on $|\pm\rangle$.

Define $S_+ = S_x + iS_y$

$S_- = S_x - iS_y$

$S_\pm$ not hermitian! $\quad S_\pm^\dagger = S_\mp$

Exercise: show $[S_z, S_\pm] = \pm\hbar S_\pm$ [few lines of algebra]

show $S_z S_\pm = S_\pm (S_z \pm \hbar)$

Let $|m\rangle$ be a normalized eigenstate of S_z w/ eigenvalue $m\hbar$ [note: we omit the $\hbar$]

Then $S_z S_\pm |m\rangle = S_\pm (m\hbar \pm \hbar)|m\rangle$

$= (m \pm 1)\hbar \; S_\pm |m\rangle$

ie $S_\pm |m\rangle$ is an eigenstate of S_z w/ eigenvalue $(m\pm1)\hbar$

S_+ is a raising operator $\quad S_+|m\rangle \sim |m+1\rangle$

S_- is a lowering operator $\quad S_-|m\rangle \sim |m-1\rangle$

$S_\pm|m\rangle$ are not necessarily normalized so

$$S_+|m\rangle = c_+(m)\,|m+1\rangle$$

$$S_-|m\rangle = c_-(m)\,|m-1\rangle$$

Consider spin-$\frac{1}{2}$.

only 2 eigenstates of S_z $\begin{cases} |+\rangle = |\tfrac{1}{2}\rangle \\ |-\rangle = |-\tfrac{1}{2}\rangle \end{cases}$

$S_+|+\rangle = 0$ because $|\tfrac{3}{2}\rangle$ doesn't exist $\Rightarrow c_+(\tfrac{1}{2}) = 0$

$S_-|-\rangle = 0$ because $|-\tfrac{3}{2}\rangle$ doesn't exist $\Rightarrow c_-(-\tfrac{1}{2}) = 0$

$S_-|+\rangle = c_-(\tfrac{1}{2})\,|-\rangle$ [only 2 left]

$S_+|-\rangle = c_+(-\tfrac{1}{2})\,|+\rangle$

Let's calculate c's:

$\langle -|\, c_-^*(\tfrac{1}{2}) = \langle +|\, S_-^\dagger = \langle +|\, S_+$

Next consider

$$\langle -|\, c_-^*(\tfrac{1}{2})\, c_-(\tfrac{1}{2})\, |-\rangle = \langle +|\, S_+ S_- \,|+\rangle$$

<u>Exercise</u>: show $S_+S_- = S_x^2 + S_y^2 + \hbar S_z$

$$= S^2 - S_z^2 + \hbar S_z$$

where $S^2 \equiv S_x^2 + S_y^2 + S_z^2$

$$\Rightarrow \quad |c_-(\tfrac{1}{2})|^2 = \langle +|\, S^2 - S_z^2 + \hbar S_z \,|+\rangle$$

$$= \langle +|S^2|+\rangle - \frac{\hbar^2}{4} + \hbar\left(\frac{\hbar}{2}\right)$$

$$= \langle +|\, S^2 \,|+\rangle + \frac{\hbar^2}{4}$$

<u>Exercise</u>: show $S_- S_+ = S^2 - S_z^2 - \hbar S_z$

$$\Rightarrow S^2|+\rangle = (S_-S_+ + S_z^2 + \hbar S_z)\,|+\rangle$$

$$= \left(0 + \frac{\hbar^2}{4} + \frac{\hbar}{2}\right)|+\rangle$$

$$= \frac{3\hbar^2}{4}|+\rangle$$

Similarly $S^2|-\rangle = (S_+S_- + S_z^2 - \hbar S_z)\,|-\rangle$

$$= \left(0 + \frac{\hbar^2}{4} - \hbar(-\tfrac{\hbar}{2})\right)|-\rangle$$

$$= \frac{3\hbar^2}{4}|+\rangle$$

$|\pm\rangle$ are also eigenstates of S^2 w/ eigenvalue $\frac{3}{4}\hbar^2$

i.e. S^2 + S_z are compatible [not obvious, because $|\pm\rangle$ <u>not</u> eigenstates of S_x + S_y]

$$|c_-(\tfrac{1}{2})|^2 = \frac{3\hbar^2}{4} + \frac{\hbar^2}{4} = \hbar^2$$

$$c_-(\tfrac{1}{2}) = \hbar e^{i\theta} \quad \Rightarrow \quad S_-|+\rangle = \hbar e^{i\theta}|-\rangle$$

What about $c_+(-\frac{1}{2})$?

$$S_+|-\rangle = c_+(-\tfrac{1}{2})|+\rangle$$

$$c_+(-\tfrac{1}{2}) = \langle +|S_+|-\rangle = \langle -|c_-^*(\tfrac{1}{2})|-\rangle = c_-^*(-\tfrac{1}{2})$$

$$c_+(-\tfrac{1}{2}) = \hbar e^{-i\theta} \quad \Rightarrow \quad S_+|-\rangle = \hbar e^{-i\theta}|+\rangle$$

Redefine $|-\rangle' = e^{i\theta}|-\rangle \Rightarrow$ $S_-|+\rangle = \hbar|-\rangle'$

$S_+|-\rangle' = \hbar|+\rangle$

Drop primes: $\boxed{\begin{aligned} S_-|+\rangle &= \hbar|-\rangle \\ S_+|-\rangle &= \hbar|+\rangle \end{aligned}}$

$$S_x = \frac{1}{2}(S_+ + S_-) \Rightarrow \quad \begin{aligned} S_x|+\rangle &= \frac{\hbar}{2}|-\rangle \\ S_x|-\rangle &= \frac{\hbar}{2}|+\rangle \end{aligned}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) \Rightarrow \quad \begin{aligned} S_y|+\rangle &= \frac{i\hbar}{2}|-\rangle \\ S_y|-\rangle &= -\frac{i\hbar}{2}|+\rangle \end{aligned}$$

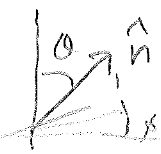
$$\langle \pm | S_x | \pm \rangle = \langle \pm | \frac{\hbar}{2} | \mp \rangle = 0$$

[makes sense: avg value of S_x for spin up/down = 0]

$$\langle \pm | S_y | \pm \rangle = 0$$

$$\vec{S} = (S_x, S_y, S_z)$$

$$\langle \pm | \vec{S} | \pm \rangle = (0, 0, \pm \tfrac{\hbar}{2}) = \pm \frac{\hbar}{2} \hat{z}$$



Let $\hat{n} = (n_x, n_y, n_z) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$

Define $S_n = \hat{n} \cdot \vec{S}$ = component of $\vec{S}$ in $\hat{n}$ direction

$$= \sin\theta \cos\phi \, S_x + \sin\theta \sin\phi \, S_y + \cos\theta \, S_z$$

$$\langle \pm | S_n | \pm \rangle = \pm \frac{\hbar}{2} \hat{n} \cdot \hat{z} = \pm \frac{\hbar}{2} \cos\theta$$