Hermitian conjugate (generalization of complex conjugate)

Given an operator $\hat{A}$, there exists another operator $\hat{A}^\dagger$ called the hermitian conjugate of $\hat{A}$.

(analog of complex conjugate of a number)

If $\quad \hat{A}\,|\psi\rangle = |\chi\rangle \quad$ then $\quad \langle\chi| = \langle\psi|\,\hat{A}^\dagger$

(this defines $\hat{A}^\dagger$).

$$\Rightarrow \quad \langle\psi|\,\hat{A}^\dagger\,|\phi\rangle = \langle\chi|\phi\rangle = \langle\phi|\chi\rangle^* = \langle\phi|\,\hat{A}\,|\psi\rangle^*$$

- $(\hat{A}^\dagger)^\dagger = \hat{A}$ $\quad$ (proof: $\langle\psi|(\hat{A}^\dagger)^\dagger|\phi\rangle = \langle\phi|\hat{A}^\dagger|\psi\rangle^* = \langle\psi|\hat{A}|\phi\rangle$ for any ϕ, ψ)
- If $\hat{A} = \alpha\,\hat{\mathbb{1}}$, then $\hat{A}^\dagger = \alpha^*\,\hat{\mathbb{1}}$
- $(\hat{B}\hat{A})^\dagger = \hat{A}^\dagger\hat{B}^\dagger$

  Proof: let $|\omega\rangle = \hat{B}\hat{A}\,|\psi\rangle$. Then $\langle\omega| = \langle\psi|\,(\hat{B}\hat{A})^\dagger$

  Now $|\omega\rangle = \hat{B}(\hat{A}\,|\psi\rangle) = \hat{B}\,|\chi\rangle \Rightarrow \langle\omega| = \langle\chi|\,\hat{B}^\dagger$

  but $|\chi\rangle = \hat{A}|\psi\rangle \Rightarrow \langle\chi| = \langle\psi|\hat{A}^\dagger$

  $\Rightarrow \langle\omega| = (\langle\psi|\hat{A}^\dagger)\,\hat{B}^\dagger \Rightarrow (\hat{B}\hat{A})^\dagger = \hat{A}^\dagger\hat{B}^\dagger$

- $\langle\phi|\,\hat{A}\hat{B}\,|\psi\rangle^* = \langle\psi|\,\hat{B}^\dagger\hat{A}^\dagger\,|\phi\rangle$

$\Rightarrow$ c.c. of a bracket is obtained by: $\quad$ ket $\leftrightarrow$ bra

$\hat{A} \leftrightarrow \hat{A}^\dagger$

reverse the order

An operator $\hat{A}$ is hermitian if $\hat{A}^\dagger = \hat{A}$
(anti hermitian if $\hat{A}^\dagger = -\hat{A}$).

An observable A is associated w/ a <u>hermitian</u> operator $\hat{A}$

Hermiticity $\Rightarrow$ $\begin{cases} \text{reality of eigenvalues} \\ \text{orthogonality of eigenstates} \end{cases}$

e.g. energy E is associated w/ the Hamiltonian $\hat{H}$
(cf. classical mechanics)

$\hat{H}|E_n\rangle = E_n|E_n\rangle \qquad n = 1, 2, \ldots \infty$ [∞ dim'l space; assume denumerable]

$|E_n\rangle$ are states of determinate energy E_n

Assume $|E_n\rangle$ are normalized $\langle E_n|E_n\rangle = 1$

$\langle E_n|\hat{H}|E_n\rangle = E_n$

$\langle E_n|\hat{H}^\dagger|E_n\rangle = \langle E_n|\hat{H}|E_n\rangle^* = E_n^*$

but $\hat{H}^\dagger = \hat{H}$ so $E_n^* = E_n$ (real eigenvalues)

Eigenstate corresponding to <u>distinct</u> eigenvalue are orthogonal

Consider $\langle E_m | \hat{H} | E_n \rangle = E_n \langle E_m | E_n \rangle$

Take c.c $\langle E_n | \hat{H}^\dagger | E_m \rangle = E_m^* \langle E_n | E_m \rangle$

$H^\dagger = H \Rightarrow \langle E_n | \hat{H} | E_m \rangle = E_n \langle E_n | E_m \rangle$

But $\langle E_n | \hat{H} | E_m \rangle = E_m \langle E_n | E_m \rangle$

$\Rightarrow (E_n - E_m) \langle E_n | E_m \rangle = 0$ so $E_n \neq E_m \Rightarrow \langle E_n | E_m \rangle = 0$

if eigenstate degenerate, proof fails,
but an orthogonal basis can always be chosen.
E.g assume $|E_n^1\rangle$ and $|E_n^2\rangle$ both have eigenvalue E_n
and $\langle E_n^1 | E_n^2 \rangle \neq 0$. Then define

$|\tilde{E}_n^2\rangle = |E_n^2\rangle - \langle E_n^1 | E_n^2 \rangle |E_n^1\rangle \quad \Rightarrow \hat{H} |\tilde{E}_n^2\rangle = E_n |\tilde{E}_n^2\rangle$

$\langle E_n^1 | \tilde{E}_n^2 \rangle = 0$ ✓ (Gram-Schmidt orthonormalization)

$$\boxed{\text{The energy eigenstates are } \underline{\text{orthonormal}} \quad \langle E_n | E_m \rangle = \delta_{nm}}$$

$\delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$

[As we did before on 4-7]

We will assume that the eigenstates of the Hamiltonian are <u>complete</u>, that is they form a basis (orthonormal) for the Hilbert space

Any $|\psi\rangle$ can be written

$$|\psi\rangle = \sum c_n |E_n\rangle$$

↑ coefficients of $|\psi\rangle$ in the energy basis

then

$$c_n = \langle E_n|\psi\rangle$$

Proof: $\langle E_n|\psi\rangle = \langle E_n|\left(\sum c_m |E_m\rangle\right) = \sum c_m \langle E_n|E_m\rangle = \sum c_m \delta_{nm} = c_n$

Thus

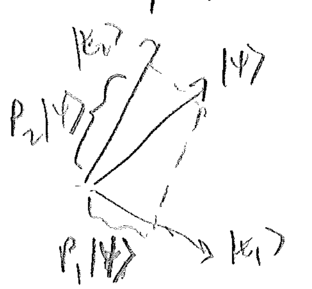
$$|\psi\rangle = \sum \langle E_n|\psi\rangle\,|E_n\rangle = \sum |E_n\rangle\langle E_n|\psi\rangle$$

$$= \left(\sum |E_n\rangle\langle E_n|\right)|\psi\rangle$$

$$\Rightarrow \boxed{\sum_n |E_n\rangle\langle E_n| = \mathbb{1} \quad \text{completeness relation}}$$

[we did this before for spin]

Define projection operators $P_n = |E_n\rangle\langle E_n|$



$P_n|\psi\rangle = c_n|E_n\rangle$ projects $|\psi\rangle$ onto $|E_n\rangle$ component

$$\begin{cases} P_n P_m = 0 \text{ if } n \neq m \\ P_n^2 = P_n \\ \sum_n P_n = 1 \end{cases}$$

$$\hat{H} = \sum E_n P_n = \sum E_n |E_n\rangle\langle E_n|$$

[show]

If measure E, probability of obtaining E_n is

$$p_n = |c_n|^2 = |\langle E_n|\psi\rangle|^2 = \langle E_n|\psi\rangle^* \langle E_n|\psi\rangle = \langle\psi|E_n\rangle\langle E_n|\psi\rangle$$

provided $|\psi\rangle$ is normalized.

Then $\sum p_n = \sum \langle\psi|E_n\rangle\langle E_n|\psi\rangle = \langle\psi|\Big(\underbrace{\sum_n |E_n\rangle\langle E_n|}_{\mathbb{1}}\Big)|\psi\rangle = \langle\psi|\psi\rangle = 1$

After E_n is measured, the state collapses to $|E_n\rangle$

$$|\psi\rangle \xrightarrow[\text{if measure } E_n]{} |E_n\rangle \qquad \text{[discontinuous]}$$

Expectation value (revisited)

$$\langle E\rangle = \sum p_n E_n = \langle\psi|\hat{H}|\psi\rangle$$

$$\langle E^2\rangle = \langle\psi|\hat{H}^2|\psi\rangle$$

uncertainty

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\langle\psi|(\hat{H}-\langle E\rangle)^2|\psi\rangle}$$

problem

① Physical definition of energy eigenstate: state of definite energy ∴ zero uncertainty $\Delta E = 0$

② Math def: $\hat{H}|E_n\rangle = E_n|E_n\rangle$

② → ① is easy $\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\underbrace{\langle E_n|\hat{H}^2|E_n\rangle}_{E_n^2} - \underbrace{\langle E_n|\hat{H}|E_n\rangle^2}_{E_n^2}} = 0$

① → ② $0 = \Delta E = \sqrt{\langle\psi|(\hat{H} - \langle E\rangle)^2|\psi\rangle}$

$$0 = \langle\psi|\,(\hat{H} - \langle E\rangle)(\hat{H} - \langle E\rangle)\,|\psi\rangle$$

Define $\hat{\Delta}_E = \hat{H} - \langle E\rangle$. $\hat{\Delta}_E^\dagger = \hat{\Delta}_E$

$$0 = \langle\psi|\,\hat{\Delta}_E^\dagger\,\hat{\Delta}_E\,|\psi\rangle$$

Define $|\chi\rangle = \hat{\Delta}_E|\psi\rangle \Rightarrow \langle\chi| = \langle\psi|\hat{\Delta}_E^\dagger$

$0 = \langle\chi|\chi\rangle \Rightarrow |\chi\rangle = 0$

$$\hat{\Delta}_E|\psi\rangle = 0$$

$$(\hat{H} - \langle E\rangle)|\psi\rangle = 0$$

$$\hat{H}|\psi\rangle = \langle E\rangle|\psi\rangle$$

$\Rightarrow |\psi\rangle$ is a mathematical eigenstate