OPERATORS and EIGENVALUES

An operator $\hat{A}$ acts on a ket to give another ket

$$\hat{A}|\psi\rangle = |\chi\rangle$$

An operator is <u>linear</u> if

$$\hat{A}\left[|\psi\rangle + |\phi\rangle\right] = \hat{A}|\psi\rangle + \hat{A}|\phi\rangle$$

and $\hat{A}\left[\alpha|\psi\rangle\right] = \alpha\hat{A}|\psi\rangle$

Eigenstate of an operator (math def.)

If $\hat{A}|\psi\rangle$ produces a ket proportional to $|\psi\rangle$

ie $\hat{A}|\psi\rangle = \lambda|\psi\rangle$

then $|\psi\rangle$ is called an eigenstate/eigenket/eigenvector of $\hat{A}$ and λ is its associated eigenvalue

- If $|\psi\rangle$ is an eigenstate, so is $\alpha|\psi\rangle$ for $\alpha \in \mathbb{C}$ [due to linearity]
- If ~~more than~~ only one independent eigenstate corresponds to ~~the same~~ a given eigenvalue λ ~~then the~~ that eigenstate is called "nondegenerate"
- If several indep eigenstates correspond to same eigenvalue λ degenerate : # of indep eigenstates = degree of degeneracy

[math meets physics]

$\boxed{\text{observables} \longleftrightarrow \text{operators}}$

Associated with each observable A
is a linear, hermitian operator $\hat{A}$

whose eigenstates are state vectors of definite A.
and whose eigenvalues are the possible results of measurement of A

$\hat{A}|a_n\rangle = a_n|a_n\rangle$ [definition of hermitian later]

(ie phys. def = math def.)

Since $|+\rangle$ and $|-\rangle$ are states of definite S_z,

$$\hat{S}_z |+\rangle = \frac{\hbar}{2} |+\rangle$$

$$\hat{S}_z |-\rangle = -\frac{\hbar}{2} |-\rangle$$

Note: states of indefinite S_z are not eigenstates of $\hat{S}_z$

$$\begin{aligned} \hat{S}_z |\psi\rangle &= \hat{S}_z \left(\alpha|+\rangle + \beta|-\rangle \right) \\ &= \alpha \hat{S}_z |+\rangle + \beta \hat{S}_z |-\rangle \quad \text{(by linearity)} \\ &= \alpha \frac{\hbar}{2} |+\rangle + \beta\left(-\frac{\hbar}{2}\right)|-\rangle \quad \text{(eigenvectors)} \\ &= \frac{\hbar}{2}\left(\alpha|+\rangle - \beta|-\rangle \right) \end{aligned}$$

Q: is $|\psi\rangle$ an eigenstate of $\hat{S}_z$?

Iff $\alpha = 0$ or $\beta = 0$

$\boxed{\text{Inner product}}$ [There is a "dual vector space" isomorphic to state space] 5-3
(↦ not necessary if no dual) (cf Cohen Tannoudji)

Associated w/ each ket $|\psi\rangle$ in the vector space of states is a "bra" $\langle\psi|$ in the dual vector space (left half of bracket)

[eg $z \to z^*$]

Combine $\langle\phi|$ w/ $|\psi\rangle$ to get bracket $\langle\phi|\psi\rangle$
which is a complex number

"inner product of $|\psi\rangle$ + $|\phi\rangle$"

[like a dot product]

Postulate
① inner product is linear

$$\langle\psi|\left[c|\phi\rangle\right] = c\langle\psi|\phi\rangle$$

$$\langle\psi|\left[|\phi_1\rangle + |\phi_2\rangle\right] = \langle\psi|\phi_1\rangle + \langle\psi|\phi_2\rangle$$

② Symmetric: $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$ [almost sym'c like dot product but not quite]

(Note $\langle\psi|\psi\rangle = \langle\psi|\psi\rangle^*$ ie $\langle\psi|\psi\rangle$ is real.)

③ non-negative: $\langle\psi|\psi\rangle > 0$ for $|\psi\rangle \neq 0$

If $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ then $\langle\psi| = \alpha^*\langle+| + \beta^*\langle-|$

Proof: for any $\langle\phi|$,

$$\langle\phi|\psi\rangle = \alpha\langle\phi|+\rangle + \beta\langle\phi|-\rangle \quad \text{(linearity i.p.)}$$

$$\langle\phi|\psi\rangle^* = \alpha^*\langle\phi|+\rangle^* + \beta^*\langle\phi|-\rangle^* \quad \text{(c.c. both sides)}$$

$$\langle\psi|\phi\rangle = \alpha^*\langle+|\phi\rangle + \beta^*\langle-|\phi\rangle \quad \text{(postulate ②)}$$

$$\langle\psi| = \alpha^*\langle+| + \beta^*\langle-| \quad \text{(because valid for any } |\phi\rangle)$$

$\boxed{\text{Normalization}}$

$\sqrt{\langle\psi|\psi\rangle}$ is called the <u>norm</u> of $|\psi\rangle$

If $\langle\psi|\psi\rangle = 1$, then $|\psi\rangle$ is <u>normalized</u>

If $\langle\psi|\psi\rangle = N$ then define $|\psi'\rangle = \frac{1}{\sqrt{N}}|\psi\rangle$

thus $\langle\psi'|\psi'\rangle = \frac{1}{N}\langle\psi|\psi\rangle = 1$

This is called "normalizing $|\psi\rangle$".

$\boxed{\text{orthogonality}}$

If $\langle\phi|\psi\rangle = 0$ then $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

$\langle\phi|\psi\rangle$ is a bracket (= complex #)

$|\psi\rangle\langle\phi|$ is an operator:

it acts on $|\chi\rangle$ to give

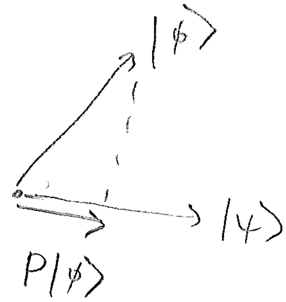
$$(|\psi\rangle\langle\phi|)\,|\chi\rangle = |\psi\rangle\langle\phi|\chi\rangle = \langle\phi|\chi\rangle\,|\psi\rangle$$

If $|\psi\rangle$ is normalized then

$|\psi\rangle\langle\psi|$ is a projection operator P

(which projects any ~~ket~~ $|\phi\rangle$ onto $|\psi\rangle$

and $|\psi\rangle$ onto itself)



[if $|\phi\rangle \perp |\psi\rangle$, then $P|\phi\rangle = 0$]

$$P^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|)$$
$$= \underbrace{\langle\psi|\psi\rangle}_{1}\;\underbrace{|\psi\rangle\langle\psi|}_{P}$$

$$P^2 = P$$

Assume that S_z eigenstates $|+\rangle$ and $|-\rangle$
are normalized

$$\langle +|+\rangle = 1$$

$$\langle -|-\rangle = 1$$

[if not, normalize them]

We'll prove later that they are orthogonal

$$\langle +|-\rangle = 0$$

Since we assumed that eigenstates are complete, i.e. span the space
~~Then~~ $|+\rangle$ and $|-\rangle$ form an $\boxed{\text{ORTHONORMAL BASIS}}$
for the space of states.

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

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The coefficients of a state in an orthonormal basis are given by inner product of the state w/ the corresponding eigenbra

i.e.: $\alpha = \langle +|\psi\rangle$

$\beta = \langle -|\psi\rangle$

Proof $\langle +|\psi\rangle = \langle +|\left(\alpha|+\rangle + \beta|-\rangle\right)$ linearity

$= \alpha\langle +|+\rangle + \beta\langle +|-\rangle$ orthonormality

$= \alpha$

$$\begin{aligned}
|\psi\rangle &= |+\rangle\alpha + |-\rangle\beta \\
&= |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle \\
&= |+\rangle\langle +|\,|\psi\rangle + |-\rangle\langle -|\,|\psi\rangle
\end{aligned}$$

$$\mathbb{1}|\psi\rangle = \left[|+\rangle\langle +| + |-\rangle\langle -|\right]|\psi\rangle$$

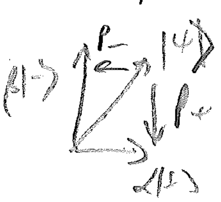
$$\mathbb{1} = |+\rangle\langle +| + |-\rangle\langle -| \quad \text{(completeness relation)}$$

$P_+ = |+\rangle\langle +|$ $\Rightarrow$ $P_+ + P_- = \mathbb{1}$

$P_- = |-\rangle\langle -|$

In particular $P_+|\psi\rangle = |+\rangle\langle +|\psi\rangle = \alpha|+\rangle$

so P_+ is the projection operator onto $|+\rangle$ (measurement)



Also, $\langle\psi|P_+|\psi\rangle = \alpha\langle\psi|+\rangle = \alpha\langle +|\psi\rangle^* = \alpha\alpha^* = |\alpha|^2$

$\langle\psi|P_-|\psi\rangle = |\beta|^2$