Isolated two body problem w/ conservative central force

reduced to one body problem in a plane (plus cm motion)

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 - U(r)$$

2 conserved quantities associated w/ invariance of L under ϕ and t

$$\frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad \ell = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}$$

$$\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad E = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + U(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{U^{eff}(r)}$$

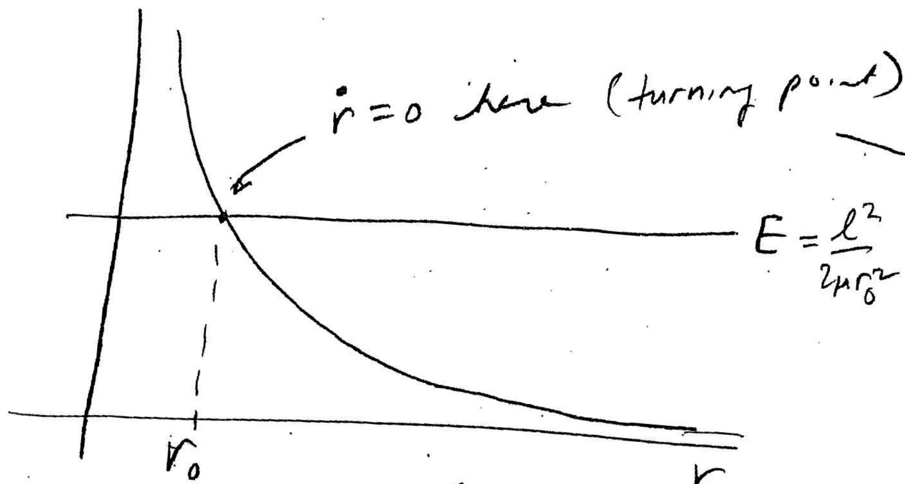
$$\boxed{U^{eff} = \frac{\ell^2}{2\mu r^2} + U(r)}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + U^{eff}(r)$$

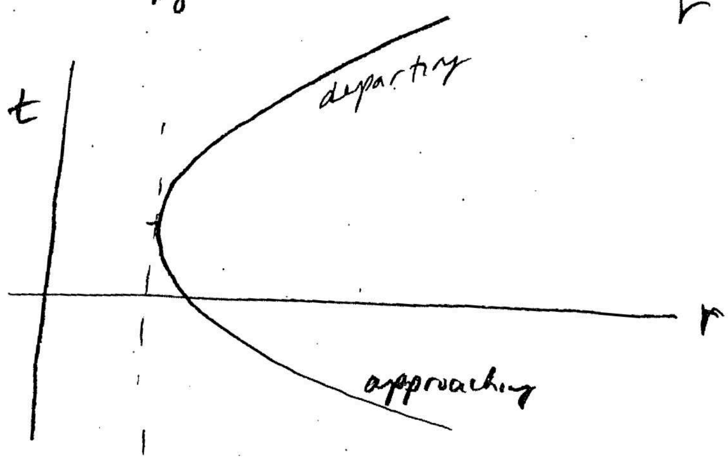
g^2

First suppose there is no physical force ($U = 0$)

$$U^{eff} = \frac{\ell^2}{2\mu r^2}$$

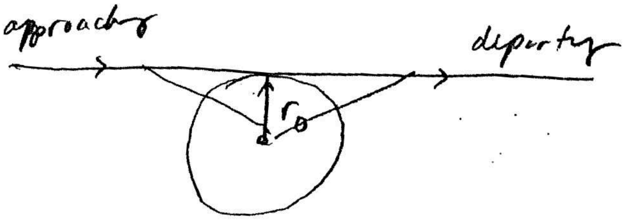


→ but not at rest because there is motion in azimuthal direction!



From a 1d viewpoint there is a repulsive radial force

From a 2d viewpoint, no force



$$\ell = \mu v_0 b$$

$$E = \frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu\left(\frac{\ell}{\mu b}\right)^2 = \frac{\ell^2}{2\mu b^2}$$

Suppose there is an attractive power law force ($k>0$)

$$F_r = -kr^{n-1} \quad \Rightarrow \quad U = \frac{kr^n}{n} \qquad \left(F_r = -\frac{dU}{dr}\right)$$

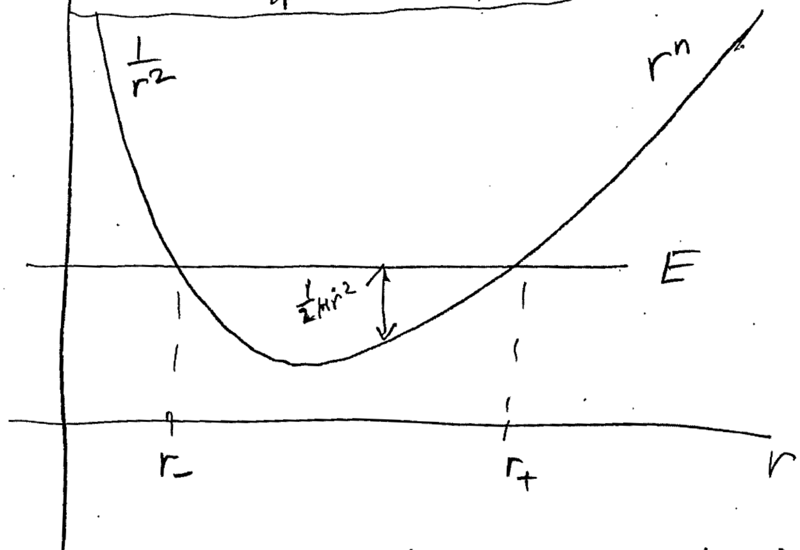
← (recall virial thm.)

e.g.

n	F_r	U	
2	$-kr$	$\frac{1}{2}kr^2$	(Hooke's Law)
1	$-k$	kr	(constant) strong force for quarks
0	$-\frac{k}{r}$	$k \log r$	
-1	$-\frac{k}{r^2}$	$-\frac{k}{r}$	(Inverse square)

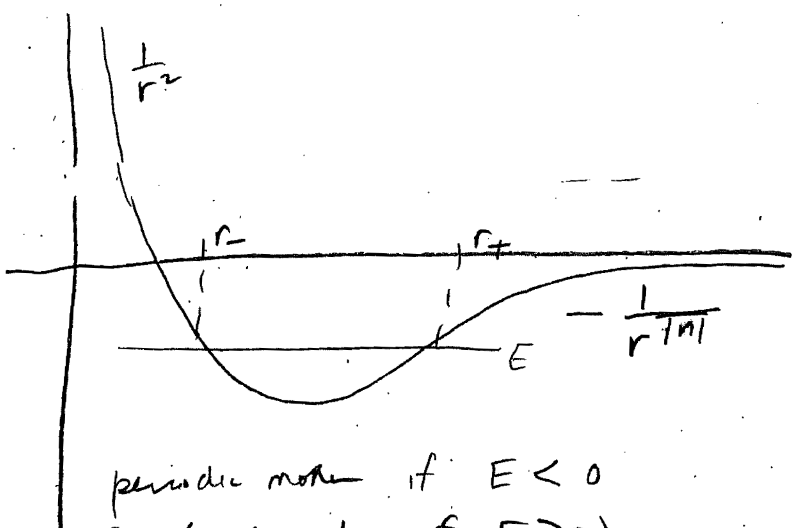
For $n > 0$
$(k > 0)$

$$\boxed{U_{eff} = \frac{\ell^2}{2\mu r^2} + \frac{k r^n}{n}}$$



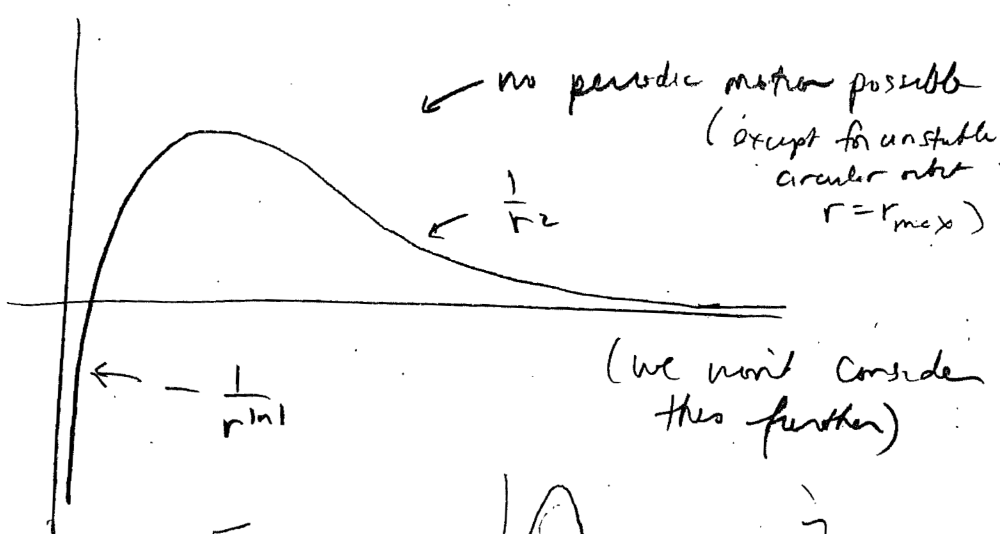
(periodic motion between r_- and r_+)

For $-2 < n < 0$
$(k > 0)$

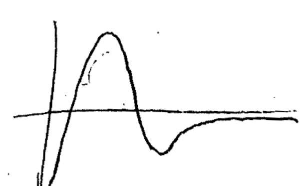


periodic motion if $E < 0$
(unbound motion if $E > 0$)

For $n < -2$
$(k > 0)$



(we won't consider this further)

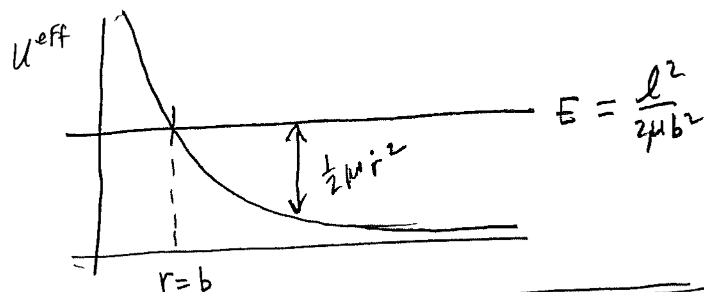
[Black hole ]

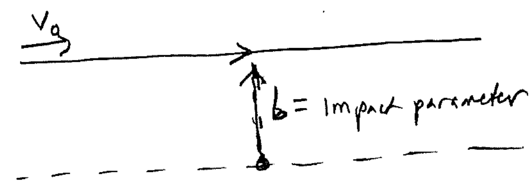
Perhaps omit this ← skipped in 2024

Let $-2 < n < 0$ (including $n = -1$: Coulomb force)

$$U^{eff} = \frac{l^2}{2\mu r^2} + \frac{k r^n}{n}$$

First : $k = 0$ (no force)

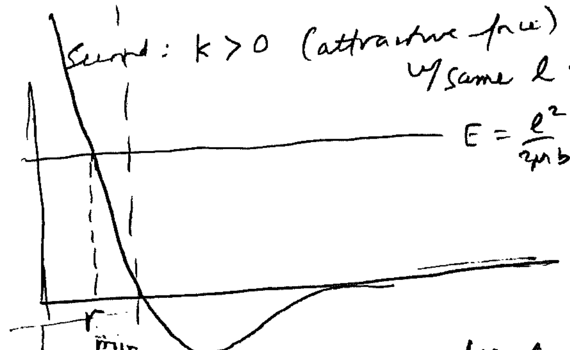




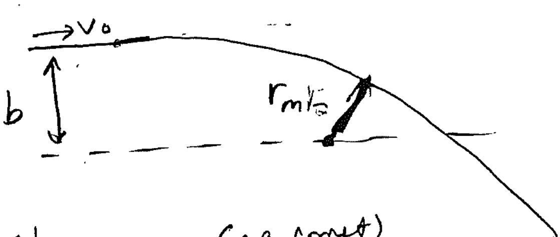
$l = \mu v_0 b$

$E = \frac{(\mu v_0 b)^2}{2\mu b^2} = \frac{1}{2}\mu v_0^2$ ✓

Second : $k > 0$ (attractive force) w/ same l + E ⇒ same v_0 + b

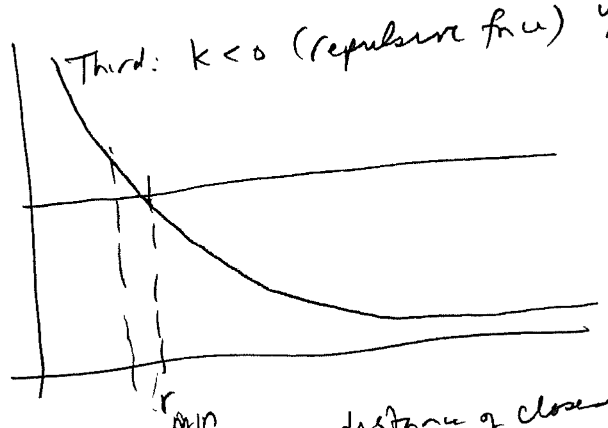


distance of closest approach $r_{min} < b$

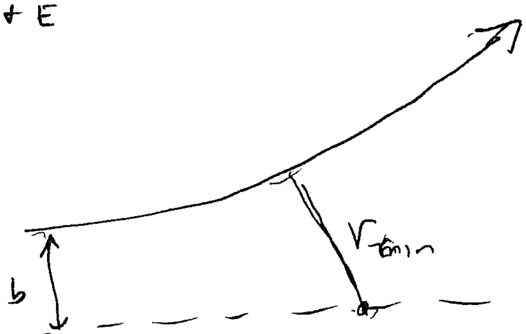


(e.g. comet)

Third: $k < 0$ (repulsive force) w/ same l + E



distance of closest approach $r_{min} > b$



(e.g. α particle scattering)

Consider $U(r)$ more general than $\frac{k}{n} r^n$?

Under what conditions does U^{eff} have a minimum at $r = r_0$?

$$U^{eff} = \frac{\ell^2}{2\mu r^2} + U(r)$$

$$\frac{dU^{eff}}{dr} = -\frac{\ell^2}{\mu r^3} + U'(r)$$

$$\left.\frac{dU^{eff}}{dr}\right|_{r_0} = 0 \quad \Rightarrow \quad U'(r_0) = \frac{\ell^2}{\mu r_0^3}$$

For $U(r) = \frac{k}{n} r^n$
$k r_0^{n-1} = \frac{\ell^2}{\mu r_0^3}$

$$\frac{d^2U^{eff}}{dr^2} = +\frac{3\ell^2}{\mu r^4} + U''(r)$$

$$\left.\frac{d^2U^{eff}}{dr^2}\right|_{r_0} = \frac{3\ell^2}{\mu r_0^4} + U''(r_0) = \frac{3}{r_0} U'(r_0) + U''(r_0) > 0 \text{ for minimum}$$

For $U = \frac{k}{n} r^n$, minimum $\Rightarrow$ $\frac{3}{r_0} k r_0^{n-1} + (n-1) k r_0^{n-2} > 0$

$$(n+2) k r_0^{n-2} > 0$$

minimum requires $n > -2$ as we saw.

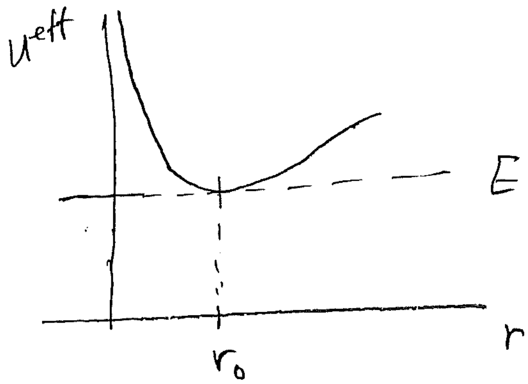
Consider stable circular orbits
(require $n > -2$)

$$\begin{cases} r = r_0 \\ \dot{r} = 0 \end{cases}$$

$$U^{eff} \equiv \frac{\ell^2}{2\mu r^2} + U(r)$$

$$E = \tfrac{1}{2}\mu \dot{r}^2 + U^{eff}(r)$$

$$\ell = \mu r^2 \dot{\phi} = \sqrt{\mu r_0^3 U'(r_0)}$$



$$\Rightarrow E = U^{eff}_{min} = \frac{\ell^2}{2\mu r_0^2} + U(r_0)$$

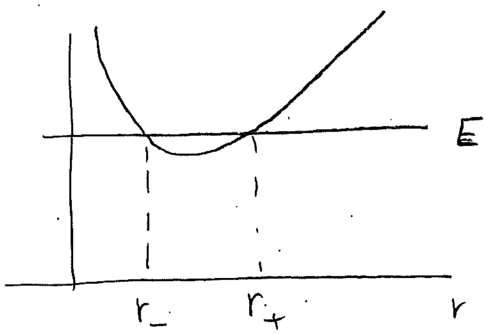
Then

$$\dot{\phi} = \frac{\ell}{\mu r_0^2}$$

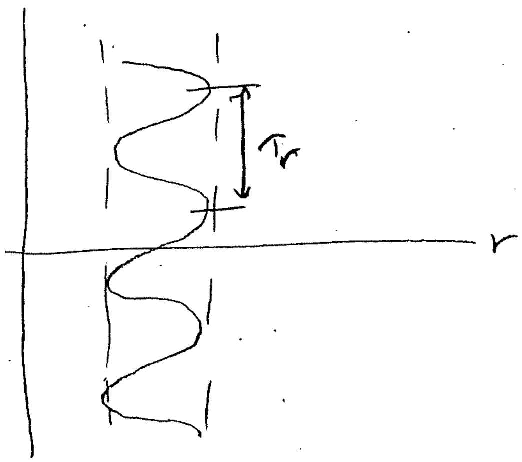
$$\phi = \frac{\ell}{\mu r_0^2} t + \phi_0 \quad \Rightarrow \quad 2\pi = \frac{\ell}{\mu r_0^2} T_\phi$$

orbital period $\displaystyle T_\phi = \frac{2\pi \mu r_0^2}{\ell} = 2\pi \left(\frac{\mu r_0}{U'(r_0)} \right)^{\frac{1}{2}}$

Radially bound orbits



Periodic radial motion between r_- + r_+
w/ radial period T_r



$$\frac{1}{2}\mu\dot{r}^2 = E - U^{eff}(r)$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U^{eff})}$$

$$t = \int dt = \underbrace{\int \frac{dr}{\sqrt{\frac{2}{\mu}(E - U^{eff}(r))}}}_{\text{solve if possible + invert to find } r(t)}$$

$$\frac{1}{2}T_r = \int_{r_-}^{r_+} \frac{dr}{\sqrt{\frac{2}{\mu}(E - U^{eff}(r))}}$$

Orbital motion $\quad \dfrac{d\phi}{dt} = \dfrac{l}{\mu r^2}$

$$\phi = \int d\phi = \int \frac{l\,dt}{\mu r^2(t)}$$ ← use explicit form of $r(t)$ if available

orbital period: $$2\pi = \int_0^{T_\phi} \frac{l\,dt}{\mu r^2(t)}$$

solve if possible to obtain T_ϕ

Nearly circular orbits

For $r \approx r_0$ we can approximate

$$U^{eff}(r) = U^{eff}(r_0) + \frac{1}{2} \underbrace{\left. \frac{d^2 U^{eff}}{dr^2} \right|_{r_0}}_{k_{eff}} \cdot (r - r_0)^2 + \dots$$

where $k_{eff} = \left. \frac{d^2 U^{eff}}{dr^2} \right|_{r_0} = \frac{3}{r_0} U'(r_0) + U''(r_0)$ [from earlier]

Radial oscillation is nearly harmonic of period

$$T_r = 2\pi \sqrt{\frac{\mu}{k_{eff}}} = 2\pi \sqrt{\frac{\mu r_0}{3U'(r_0) + r_0 U''(r_0)}} .$$

For $U(r) = \frac{k}{n} r^n$ we have $U'(r_0) = \frac{\ell^2}{\mu r_0^3} \Rightarrow k r_0^{n-1} = \frac{\ell^2}{\mu r_0^3}$

and $k_{eff} = (n+2) k r_0^{n-2} = (n+2) \frac{\ell^2}{\mu r_0^4}$

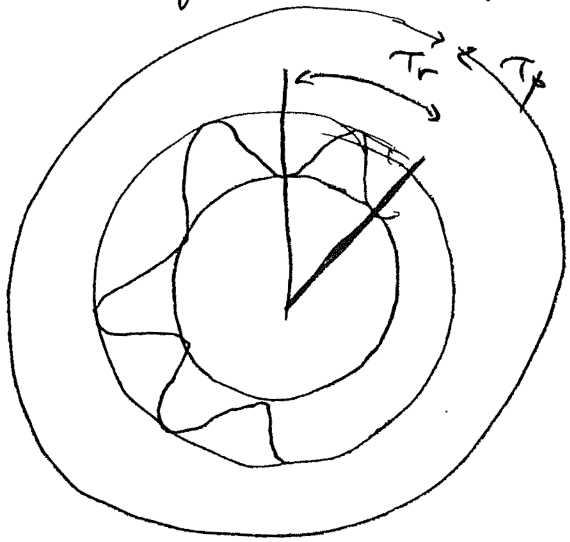
$$\Rightarrow \boxed{T_r \cong \frac{2\pi \mu r_0^2}{\ell \sqrt{n+2}}} \text{ radial period}$$

orbital period

$$2\pi = \int_0^{T_0} \frac{\ell \, dt}{\mu r^2(t)} \approx \int_0^{T_\phi} \frac{\ell \, dt}{\mu r_0^2} = \frac{\ell T_\phi}{\mu r_0^2} \Rightarrow \boxed{T_\phi \cong \frac{2\pi \mu r_0^2}{\ell}}$$

Since $U'(r_0) = \frac{\ell^2}{\mu r_0^3}$ can write $\boxed{T_\phi \cong 2\pi \sqrt{\frac{\mu r_0}{U'(r_0)}}}$

Compare ratio of (approximate) radial + orbital periods:



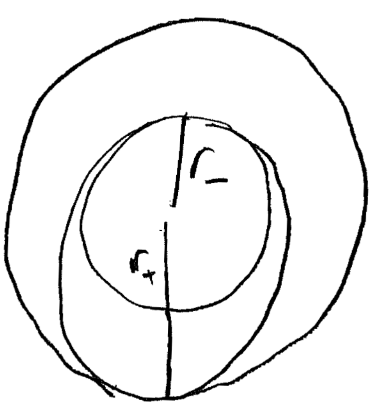
$\frac{T_\phi}{T_r}$ = number of "lobes"

$U = \frac{k}{n} r^n$

$$\frac{T_\phi}{T_r} = \sqrt{\frac{3U'(r_0) + r_0 U''(r_0)}{U'(r_0)}} = \sqrt{n+2}$$

orbit closes if $\sqrt{n+2}$ = integer (or fraction)

$n=-1$ (inverse square) : $\frac{T_\phi}{T_r} = \sqrt{-1+2} = 1 \Rightarrow T_\phi = T_r$

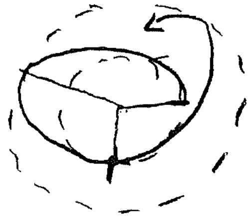


we've only shown this for approx periods

but orbit closes

→ ellipse

For $n>-1$, $\frac{T_\phi}{T_r} > 1$



orbit does not close unless
$\sqrt{n+2}$ is rational (ie integer)

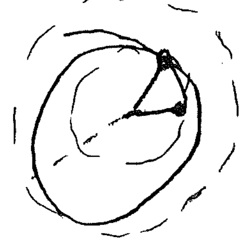
$n=2 \Rightarrow \frac{T_\phi}{T_r} = \frac{1}{2}\sqrt{2+2} = 2$

Again, although we've only shown it approximately
this result is exact

$n=2 \Rightarrow$ 2d harmonic oscillator $U = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2+y^2)$
= two 1d oscillators w/ <u>same</u> frequency!

For large n, $T_\phi \gg T_r$

For $-2<n<-1$, $\frac{T_r}{T_\phi} > 1$



Bertrand's thm: only central forces that result in closed orbits are $n=-1$ and $n=2$

M = mass of black hole + object

Thus $M\mu = (m_1+m_2)\left(\frac{m_1 m_2}{m_1+m_2}\right) = m_1 m_2$

GR

$$U^{eff} = -\frac{GM\mu}{r} - \frac{GML^2}{\mu^2 r^3} + \frac{L^2}{2\mu r^2}$$

← 275 notes

$$\boxed{\frac{T_r}{T_\phi} = 1 \pm \underbrace{\frac{3GM}{r_0 c^2}}}$$

7.6×10^{-8}

$M = 2E30$ kg

$r = 0.39$ AU $= 5.8E10$ m

$T_\phi = .24$ years

1 century (earth) = 400 mercury orbits

$(7.6\times10^{-8})\ 400 = 3\times10^{-5}$

$(3\times10^{-5})(360)(3600) = 40$ secs of arc

(2022?)

Thomas asks: for what potentials $U(r)$ do orbits close?

ie $\frac{T_r}{T_\phi} = \frac{p}{q}$ (rational #) $= C$

$$\frac{U'(r)}{3U' + rU''} = c^2$$

$$(1-3c^2)\,U' = r\,c^2\,U''$$

$$\frac{U''}{U'} = \frac{1-3c^2}{c^2}\,\frac{1}{r}$$

$$\frac{dU'}{U'} = \frac{1-3c^2}{c^2}\,\frac{dr}{r}$$

$$\ln U' = \frac{1-3c^2}{c^2}\ln r + \text{const}$$

$$U' = A r^{\frac{1-3c^2}{c^2}}$$ ← monomial

eg. $C = 1 \Rightarrow U' = A r^{-2} \Rightarrow U = \frac{-A}{r} + \text{const}$

$C = \frac{1}{2} \Rightarrow U' = A r^{1} \Rightarrow U = \frac{A r^2}{2} + \text{const}$

$C = \frac{1}{\sqrt{n+2}} \Rightarrow U' = A r^{n-1} \Rightarrow U = \frac{A r^n}{n} + \text{const}$