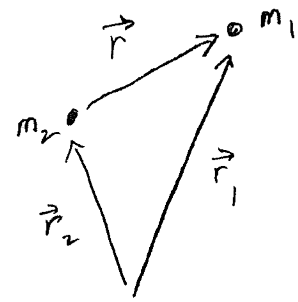
Isolated two body system

Recall from problem

$$\begin{cases} m_1 \dfrac{d^2\vec{r}_1}{dt^2} = \vec{F}^{ext}(\vec{r}_1) + \vec{F}_{12}(\vec{r}) \\ m_2 \dfrac{d^2\vec{r}_2}{dt^2} = \vec{F}^{ext}(\vec{r}_2) + \vec{F}_{21}(\vec{r}) \end{cases}$$



Isolated: $\vec{F}^{ext} = 0$

Newton's 3rd: $\vec{F}_{21} = -\vec{F}_{12} = \vec{F}$

$$\begin{cases} \dfrac{d^2 r_1}{dt^2} = \dfrac{1}{m_1}\vec{F} \\ \dfrac{d^2 r_2}{dt^2} = -\dfrac{1}{m_2}\vec{F} \end{cases}$$

Define $\begin{cases} \vec{r}_{cm} = \dfrac{m_1}{M}\vec{r}_1 + \dfrac{m_2}{M}\vec{r}_2 \\ \vec{r} = \vec{r}_1 - \vec{r}_2 \end{cases}$ $\quad M = m_1 + m_2$

$$\frac{d^2\vec{r}_{cm}}{dt} = 0 \Rightarrow \vec{r}_{cm} = \vec{V}_{cm} t + \vec{r}_{cm}(0)$$

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F} \equiv \frac{1}{\mu}\vec{F} \quad \Rightarrow \mu \frac{d^2\vec{r}}{dt^2} = \vec{F}$$

reduced mass $\quad \mu = \dfrac{1}{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)} = \dfrac{m_1 m_2}{m_1 + m_2}$

$m_1 = m_2 \Rightarrow \mu = \frac{1}{2} m_1$

$m_1 \ll m_2 \Rightarrow \mu = m_1$ (lighter)

Earlier we showed

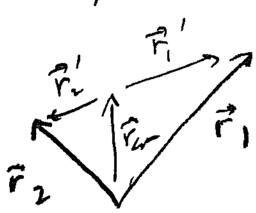
$$T^{sys} = \sum_{i=1}^{2} \frac{1}{2} m_i v_i^2 = \frac{1}{2} M v_{cm}^2 + T'^{\,sys}$$

$$\vec{L}^{sys} = \sum_{i=1}^{2} m_i \vec{r}_i \times \vec{v}_i = M \vec{r}_{cm} \times \vec{v}_{cm} + \vec{L}'^{\,sys}$$

prime = cm frame

$$T'^{\,sys} = \sum \frac{1}{2} m_i v_i'^2$$

$$\vec{L}'^{\,sys} = \sum m_i \vec{r}_i' \times \vec{v}_i'$$



$$\vec{r}_1' = \vec{r}_1 - \vec{r}_{cm} = \left(1 - \frac{m_1}{M}\right)\vec{r}_1 - \frac{m_2}{M}\vec{r}_2 = \frac{m_2}{M}(\vec{r}_1 - \vec{r}_2) = \frac{m_2}{M}\vec{r}$$

Likewise $\vec{r}_2' = \frac{m_1}{M}(\vec{r}_2 - \vec{r}_1) = -\frac{m_1}{M}\vec{r}$

$\vec{v}_1' = \frac{m_2}{M}\vec{v}$ where $\vec{v} = \frac{d\vec{r}}{dt}$

$\vec{v}_2' = -\frac{m_1}{M}\vec{v}$

Then $T'^{\,sys} = \frac{1}{2} m_1 \left(\frac{m_2}{M}\vec{v}\right)^2 + \frac{1}{2} m_2 \left(-\frac{m_1}{M}\vec{v}\right)^2$

$$= \frac{1}{2}\frac{m_1 m_2}{M}\left(\frac{m_2 + m_1}{M}\right) v^2 = \frac{1}{2}\mu v^2$$

$$\vec{L}'^{\,sys} = m_1 \left(\frac{m_2}{M}\vec{r}\right) \times \left(\frac{m_2}{M}\vec{v}\right) + m_2 \left(-\frac{m_1}{M}\vec{r}\right) \times \left(-\frac{m_1}{M}\vec{v}\right)$$

$$= \frac{m_1 m_2}{M}\left(\frac{m_2 + m_1}{M}\right)\vec{r} \times \vec{v} = \mu \vec{r} \times \vec{v}$$

$$T^{sys} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \mu v^2$$

$$E_{mech}^{sys} = \frac{1}{2} M v_{cm}^2 + \left[\frac{1}{2} \mu v^2 + U(r) \right]$$

separately conserved for isolated system
(because $\vec{v}_{cm} = $ const)

$$\frac{d}{dt}\left[\frac{1}{2}\mu v^2 + U(r)\right] = \mu \vec{v} \cdot \frac{d\vec{v}}{dt} + \vec{\nabla} U \cdot \frac{d\vec{r}}{dt}$$

$$= \vec{v} \cdot \left[\underbrace{\mu \frac{d^2\vec{r}}{dt^2} - \vec{F}}_{0} \right] = 0$$

$$L^{sys} = M \vec{r}_{cm} \times \vec{v}_{cm} + \mu \vec{r} \times \vec{v}$$

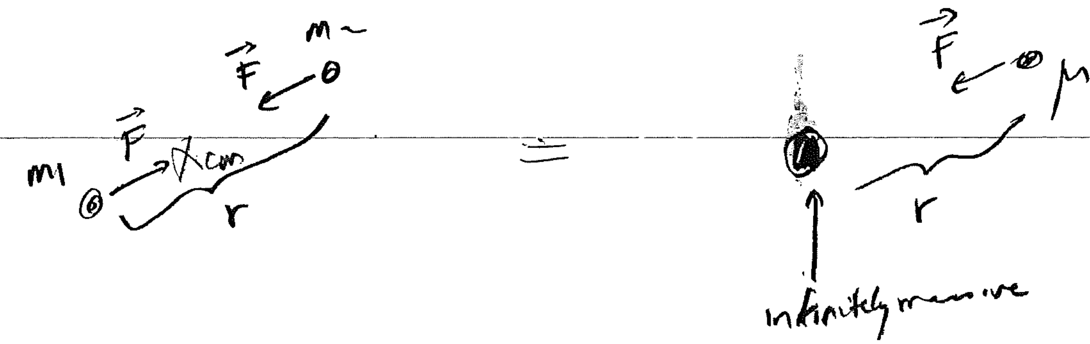
separately conserved for isolated system

$$\frac{d}{dt}(M \vec{r}_{cm} \times \vec{v}_{cm}) = \underbrace{M \vec{v}_{cm} \times \vec{v}_{cm}}_{0} + M \vec{r}_{cm} \times \underbrace{\frac{d\vec{v}_{cm}}{dt}}_{0} = 0$$

$$\frac{d}{dt}(\mu \vec{r} \times \vec{v}) = \underbrace{\mu \vec{v} \times \vec{v}}_{0} + \mu \vec{r} \times \frac{d\vec{v}}{dt} = \vec{r} \times \vec{F} = 0$$

central

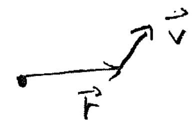
Reducing 2 body problem to 1 body



Angular momentum of one body problem

$\vec{L} = \mu \vec{r} \times \vec{v}$

$\vec{L} \perp$ to plane defined by $\vec{r}$ & $\vec{v}$



Since $\vec{L}$ conserved, $\vec{r}$ & $\vec{v}$ remain in the same plane

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Physically: $\vec{r}(0)$ & $\vec{v}(0)$ define a plane [unless parallel]

$\Delta\vec{r}$ is in direction of $\vec{v}$ so $\vec{r}$ remains in plane

$\Delta\vec{v}$ is in direction of $\pm\vec{r}$ so $\vec{v}$ remains in plane

Central force motion is planar

Choose this plane to be xy-plane

Lagrangian approach to isolated two-body problem

$$L = T - U$$

$$= \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

Instead use generalized coordinates $\vec{r}_{cm}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$L = \frac{1}{2} M \dot{\vec{r}}_{cm}^{\,2} + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - U(r)$$

$\vec{r}_{cm}$ is cyclic $\left(\frac{\partial L}{\partial \vec{r}_{cm}} = 0 \right)$ so

$$\vec{p}_{cm} = \frac{\partial L}{\partial \dot{\vec{r}}_{cm}} = M \dot{\vec{r}}_{cm} \quad \text{is const}$$

$$\vec{r}_{cm}(t) = \vec{r}_{cm}(0) + \vec{v}_{cm} t$$

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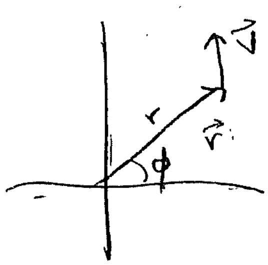
use polar coordinates for equivalent one body problem

(can restrict to xy plane due to conservation of $\vec{L}$)

$\hookrightarrow$ ($\therefore r = \rho$, $\theta = \frac{\pi}{2}$)



$$\vec{r} = r \hat{r}$$

$$\vec{v} = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi}$$

$$v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$$

$$L = \frac{1}{2} \mu \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r)$$

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

ϕ is a cyclic coordinate, so the conjugate momentum is conserved

$$\frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{const, call it } \ell$$

(note this is just z-component of angular momentum $\mu r v_\phi = \mu r^2 \dot{\phi}$)

↳ have already used other two components by restricting to xy plane

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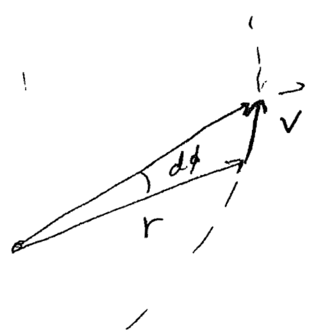
Kepler's 2nd law: equal areas swept out in equal times



$$dA = \frac{1}{2} r (r\, d\phi)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{\ell}{2\mu} = \text{const}$$

true for any central force (not just $1/r^2$)

[xkcd]

[could also write E-L eqn for r, but instead]

Since L does not depend explicitly on time

use 2nd form of E-L

$$\frac{dh}{dt} + \cancel{\frac{\partial L}{\partial t}}^{\,0} = 0 \quad \Rightarrow \quad h \text{ conserved}$$

$$h =! \ \dot r \frac{\partial L}{\partial \dot r} + \dot\phi \frac{\partial L}{\partial \dot\phi} - L$$

$$= \dot r(\mu \dot r) + \dot\phi(\mu r^2 \dot\phi) - \left(\tfrac{1}{2}\mu \dot r^2 + \tfrac{1}{2}\mu r^2 \dot\phi^2 - U(r)\right)$$

$$= \tfrac{1}{2}\mu \dot r^2 + \tfrac{1}{2}\mu r^2 \dot\phi^2 + U(r)$$

$$= T + U = E$$

But $\dot\phi = \dfrac{\ell}{\mu r^2}$ so

$$E = \tfrac{1}{2}\mu \dot r^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{U^{\text{eff}}(r)}$$

not really potential energy but rather azimuthal kinetic energy

By using cons of ang. momentum, we have converted a 3 dim'l problem w/ potential energy $U(r)$ into a one dim'l (radial) problem w/ effective potential energy $U^{\text{eff}}(r)$

2018 notes → TB suggests using 2nd form of E-L instead

Euler Lagrange eqn for radial coordinate

$$\frac{\partial L}{\partial r} = \mu r \dot{\phi}^2 - \frac{dU}{dr} \qquad \text{but} \quad \dot{\phi} = \frac{\ell}{\mu r^2}$$

$$= \frac{\ell^2}{\mu r^3} - \frac{dU}{dr}$$

$$\frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$

really $\mu a_r = \mu(\ddot{r} - r\dot{\phi}^2) = -\frac{dU}{dr}$

$$\Rightarrow \quad \mu \ddot{r} = \underbrace{\frac{\ell^2}{\mu r^3}}_{\text{"centrifugal force"}} - \underbrace{\frac{dU}{dr}}_{\text{physical force}}$$

"centrifugal force" (because we are treating a 2d problem as a 1d radial problem)

$$\mu \ddot{r} = -\frac{d}{dr}\Big(\underbrace{\frac{\ell^2}{2\mu r^2} + U}_{U^{eff}}\Big) = F_r^{eff}$$

Multiply by $\dot{r}$

$$\mu \dot{r} \ddot{r} = -\frac{dU^{eff}}{dr}\frac{dr}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right) = -\frac{d}{dt} U^{eff}$$

$$\boxed{\frac{1}{2}\mu \dot{r}^2 + U^{eff} = E}$$

use 1d potential energy diagrams

where $U^{eff} = U + \frac{\ell^2}{2\mu r^2}$

this is "really" kinetic energy in azimuthal (ϕ) direction

(9-25-17)

$\boxed{\text{Routhian}}$ Note to self

Central force in 2d

$$L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} = p \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\text{E-L} \Rightarrow \quad mr^2\dot{\theta} = l = \text{const}$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$= \frac{l^2}{mr^3} - \frac{\partial V}{\partial r} \quad ✓$$

naively

Note this is not $-V_{eff}$!

$$L = \tfrac{1}{2}m\dot{r}^2 + \overbrace{\frac{l^2}{2mr^2} - V(r)}$$

$$\frac{\partial L}{\partial r} = -\frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$m\ddot{r} = -\frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

wrong sign!!

Also $V_{eff} = V + \frac{l^2}{2mr^2}$ gives this

$$H = p\dot{r} + l\dot{\theta} - L$$

$$= \frac{p^2}{2m} + \frac{l^2}{2mr^2} + V(r)$$

$$\dot{\theta} = \frac{\partial H}{\partial l} = \frac{l}{mr^2}$$

$$\dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{l} = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot{p} = -\frac{\partial H}{\partial r} = +\frac{l^2}{mr^3} - \frac{\partial V}{\partial r} \quad ✓$$

The correct way to implement this is via the Routhian

$$R = L - l\dot{\theta}$$

This is $V_{eff}(r)$ so

$$= \tfrac{1}{2}m\dot{r}^2 \underbrace{- \frac{l^2}{2mr^2} - V(r)} \qquad R = T - V_{eff}$$

Then wrt. θ we use Hamilton's eqn's

$$\dot{\theta} = -\frac{\partial R}{\partial l} = +\frac{l}{mr^2}, \quad \dot{l} = +\frac{\partial R}{\partial \theta} = 0$$

(because I defined R as negative of the usual)

and wrt. r we use Euler-Lagrange

$$\frac{dR}{d\dot{r}} = m\dot{r}$$

$$\frac{dR}{dr} = +\frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\Rightarrow m\ddot{r} = \frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

correct sign!

I finally get the point of the Routhian!