[In 2018, I did this at end of the course]

Symmetries + conserved quantities

Symmetry $\Rightarrow$ invariance of laws of physics under an operation

$\Rightarrow$ existence of a conserved quantity (Noether's theorem) 1882–1935

In turn, the conserved quantity "generates" the sym. operation

① time translation $\Rightarrow$ cons of E

② spatial translation $\Rightarrow$ cons of $\vec{p}$

③ rotations $\Rightarrow$ cons of $\vec{L}$

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① Suppose Lagrangian is invariant, under $t \to t + t_0$

$$\left[L(q(t), \dot{q}(t), t) = L(q(t), \dot{q}(t), t + t_0) \right]$$

ie $\frac{\partial L}{\partial t} = 0$

We already derived 2nd form of E-L

$$\frac{dh}{dt} + \underbrace{\frac{\partial L}{\partial t}}_{0} = 0$$

then $\longrightarrow$ h = conserved

Define $h = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ = const

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If T is quadratic in $\dot{q}$ and U indep of $\dot{q}$, then $h = E_{mech}$

Conserv. of mechanical energy.

rev 2024

Invariance under spatial translations (homogeneity)

Consider a system of particles w/ positions $\vec{r}_i$

$$L = \sum \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - U(|\vec{r}_i - \vec{r}_j|)$$

Invariant under simultaneous shift (translation) of all particles

$$\vec{r}_i \to \vec{r}_i + \vec{n}\, a$$

$$\dot{\vec{r}}_i \to \dot{\vec{r}}_i$$

$$L \to L$$

Consider an infinitesimal shift da

$$dL = L(\vec{r}_i + \vec{n}\, da, \dot{\vec{r}}_i) - L(\vec{r}_i, \dot{\vec{r}}_i) = 0$$

$$= \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot \vec{n}\, da + O(da^2)$$

Use E-L eqn

$$= \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i}\right) \cdot \vec{n}\, da$$

$$= \frac{d}{dt}\left(\sum \vec{p}_i \cdot \vec{n}\right) da$$

$\Rightarrow \sum \vec{p}_i \cdot \vec{n} = \text{const}$ for any $\vec{n}$ $\Rightarrow$ $\underbrace{\sum \vec{p}_i}_{\vec{p}_{sys}} = \text{const}$. (momentum conservation)

Invariance under rotations (isotropy)

Recall that a vector $\vec{A}$ fixed in body-fixed frame

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$ describes its rotation in a space-fixed frame

Let $\vec{\omega} = \hat{n}\,\frac{d\theta}{dt}$ then

$$d\vec{A} = \hat{n}\, d\theta \times \vec{A}$$ describes rotation about $\hat{n}$ through $d\theta$

Consider a Lagrangian $L(\vec{r}_i, \dot{\vec{r}}_i)$ invariant under rotation

$$d\vec{r}_i = (\hat{n} \times \vec{r}_i)\, d\theta$$
$$d\dot{\vec{r}}_i = (\hat{n} \times \dot{\vec{r}}_i)\, d\theta$$
$$dL = 0$$

But

$$dL = L(\vec{r}_i + \hat{n}\times\vec{r}_i\, d\theta,\ \dot{\vec{r}}_i + \hat{n}\times\dot{\vec{r}}_i\, d\theta) - L(\vec{r}_i, \dot{\vec{r}}_i)$$

$$= \sum_i \Big[\underbrace{\frac{\partial L}{\partial \vec{r}_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i}\right)} \cdot (\hat{n}\times\vec{r}_i) + \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot (\hat{n}\times\dot{\vec{r}}_i) \Big] d\theta + O(d\theta^2)$$

$$= \sum_i \frac{d}{dt} \underbrace{\Big[\underbrace{\frac{\partial L}{\partial \dot{\vec{r}}_i}}_{\vec{p}_i} \cdot (\hat{n}\times\vec{r}_i) \Big]}_{\hat{n}\cdot(\vec{r}_i\times\vec{p}_i)} d\theta$$

$$= \frac{d}{dt}\Big(\sum_i \vec{r}_i \times \vec{p}_i \Big) \cdot \hat{n}\, d\theta = 0$$ valid for any $\hat{n}\, d\theta$

$$\Rightarrow \sum_i \vec{r}_i \times \vec{p}_i = \text{const} = \vec{L}^{sys}$$ (cons. of angular momentum)

Notes from Fields, Particles + Symmetries

Suppose there exists an infinitesimal variation of generalized coordinates δq_i + velocities $\delta \dot{q}_i$ that leaves Lagrangian invariant

$$L(q_i + \delta q_i ; \dot{q}_i + \delta \dot{q}_i) = L(q_i ; \dot{q}_i)$$

→ called a symmetry of the Lagrangian.

The variation of the Lagrangian vanishes

$$0 = \delta L = L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i) - L(q_i, \dot{q}_i)$$

$$= \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] + O(\delta q^2)$$

Use e.o.m. $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ and $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$

$$0 = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right]$$

$$= \frac{d}{dt} \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]$$

α " a infinitesimal parameter associated w/ the variation

Thus $\boxed{Q \equiv \sum \frac{\partial L}{\partial \dot{q}_i} \frac{\delta q_i}{\alpha}}$ is a conserved quantity ("charge")

[slightly different than previous case because under time translation L is not invariant.]