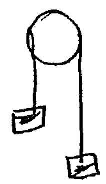
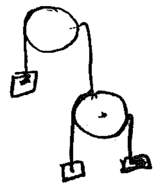
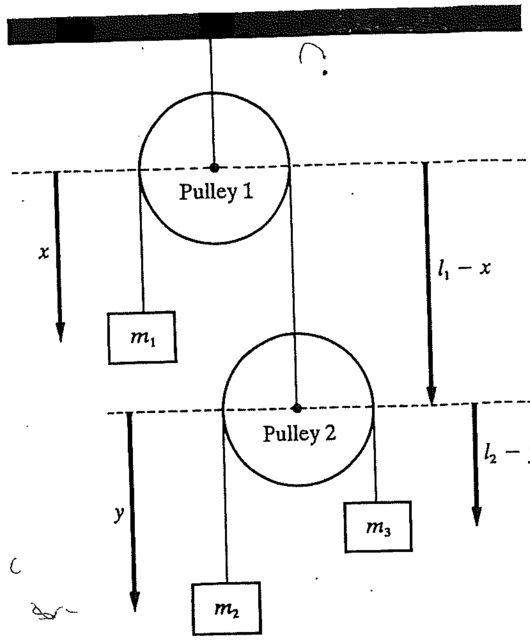
Atwood

Goldstein Taylor



Marion



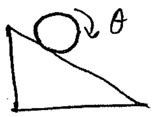


Slide down plane

Eck

Roll down plane (fixed)

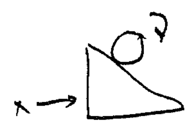
[Taylor 7-16]



frictionless plane

(Taylor w/ blocks, ex. 7.5)

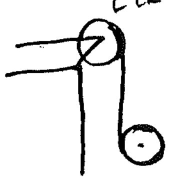
in class



problem

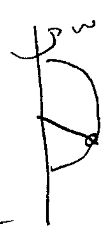
Unwinding spool

Eck

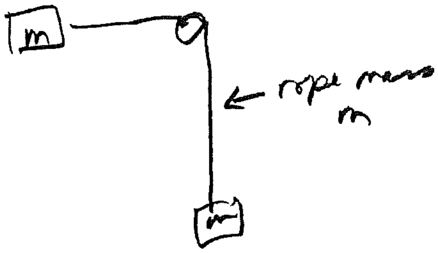


Spinning hoop

in class



(Taylor)



M+T 6-10

M+T 6-12

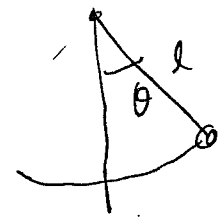
Lagrangian approach especially useful for

Constrained systems

① simple plane pendulum

(particle constrained to move on a circle)

1 d.o.f



$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r, \theta)$$

constraint $r = l \Rightarrow \dot{r} = 0$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

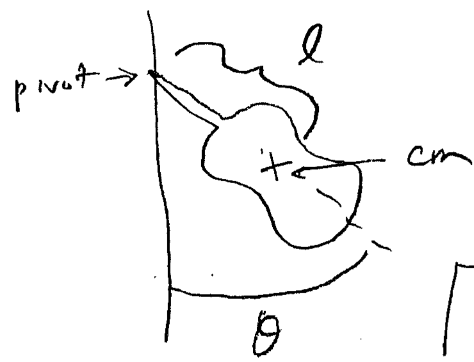
$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

Euler-Lagrange $\Rightarrow$ $ml^2\ddot{\theta} + mgl\sin\theta = 0$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$ as we found earlier

② physical pendulum

1 d.o.f.



$$L = T - U$$

$$= \frac{1}{2} I \dot{\theta}^2 + mgl\cos\theta$$

I = mom. of inertia about pivot point (fixed)

or $\frac{1}{2} I_{cm}\dot{\theta}^2 + \frac{1}{2} m v_{cm}^2$

$v_{cm} = l\dot{\theta}$

$\frac{1}{2}(I_{cm} + ml^2)\dot{\theta}^2$

Euler-Lagrange: $I\ddot{\theta} + mgl\sin\theta = 0$ as before

Observe: $\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} = -L_y$ ← angular momentum

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta = -\tau_y$ ← torque

generalized momentum

generalized force

Thus euler-lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$

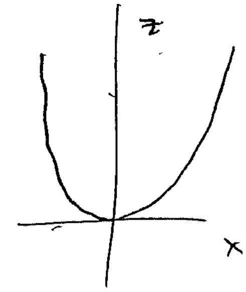
$$\Rightarrow \frac{dL_y}{dt} = \tau_y$$

Recall corollary: if L indep of θ then $\frac{\partial L}{\partial \dot{\theta}} = \text{const}$

L indep of θ $\Rightarrow$ no torque $\Rightarrow$ angular moment is conserved

[Problem: double pendulum]

③ Bead constrained to move on a fixed parabolic wire (1 d.o.f.)



Constraint: $z = cx^2$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$$

$$\dot{z} = \frac{dz}{dx}\dot{x} = 2cx\dot{x}$$

$$L = \frac{1}{2}m(1 + 4c^2x^2)\dot{x}^2 - mgcx^2$$

$$\frac{\partial L}{\partial \dot{x}} = m(1+4c^2x^2)\dot{x}, \qquad \frac{\partial L}{\partial x} = 4mc^2x\dot{x}^2 - 2mgcx$$

E-L ⇒ $$\underbrace{\frac{d}{dt}\left[m(1+4c^2x^2)\dot{x}\right]} - 4mc^2x\dot{x}^2 + 2mgcx = 0$$

$$m(1+4c^2x^2)\ddot{x} + 8mc^2x\dot{x}^2$$

$$\boxed{m(1+4c^2x^2)\ddot{x} + 4mc^2x\dot{x}^2 + 2mgcx = 0}$$

Can we integrate this? Yes. Multiply by $\dot{x}$

$$m(1+4c^2x^2)\dot{x}\ddot{x} + 4mc^2x\dot{x}^3 + 2mgcx\dot{x} = 0$$

$$= \frac{d}{dt}\left[\frac{1}{2}m(1+4c^2x^2)\dot{x}^2 + mgcx^2\right] = 0$$

How did we know this was possible?

L independent of t and T only quadratic in velocities so 2nd form of Euler's eqn gives energy conservation

$T + U = E$

solve motion $$\dot{x} = \sqrt{\frac{2(E - mgcx^2)}{m(1+4c^2x^2)}}$$

[NB. not quite a harmonic oscillator]

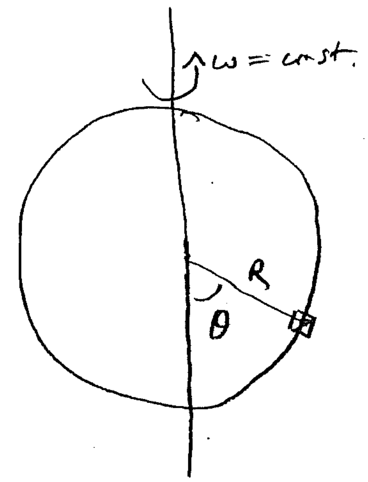
$$\int dx \sqrt{\frac{m(1+4c^2x^2)}{2(E - mgcx^2)}} = \int dt = t$$

moving constraints

e4

(4) Bead on a rotating hoop (1 d.o.f.)

[Taylor, p. 260]



$z = -R\cos\theta$

$x = R\sin\theta\cos\omega t$

$y = R\sin\theta\sin\omega t$

$\dot z = +R\sin\theta\,\dot\theta$

$\dot x = R\cos\theta\,\dot\theta\cos\omega t - \omega R\sin\theta\sin\omega t$

$\dot y = R\cos\theta\,\dot\theta\sin\omega t + \omega R\sin\theta\cos\omega t$

$$\dot x^2 + \dot y^2 + \dot z^2 = R^2\cos^2\theta\,\dot\theta^2 + \omega^2 R^2\sin^2\theta + R^2\sin^2\theta\,\dot\theta^2$$

$$= R^2\dot\theta^2 + R^2\omega^2\sin^2\theta$$

Note to self: T does *not* depend quadratically on velocities

$$L = T - U = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta$$

Euler-Lagrange: $mR^2\ddot\theta - mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta = 0$

$$\ddot\theta = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)$$

Look for equilibrium positions $\theta = \theta_0 \Rightarrow \ddot\theta = 0$

Either $\sin\theta_0 = 0 \Rightarrow \theta = 0$ or π

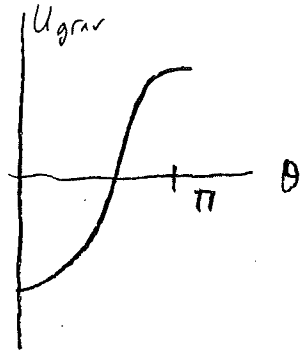
or $\cos\theta_0 = \frac{g}{\omega^2 R}$ $\Rightarrow$ possible only if $\omega^2 > \frac{g}{R}$

Stable or unstable?

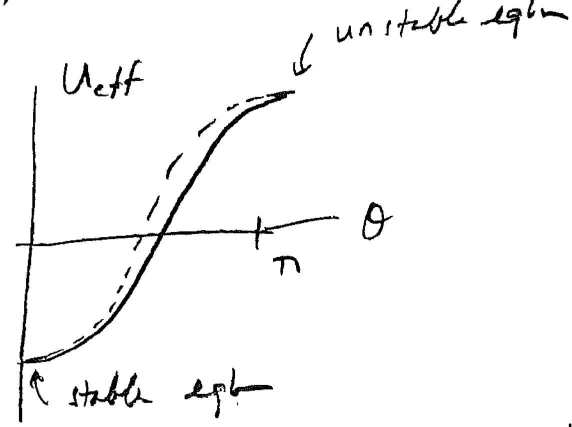
Since L doesn't depend explicitly on t,

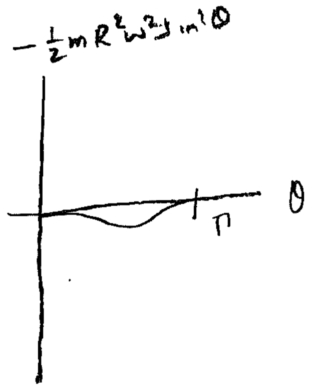
$$h = \underbrace{\dot{\theta}\frac{\partial L}{\partial \dot{\theta}}}_{mR^2\dot{\theta}^2} - L = \text{const}$$ [not $E_{mech} = T + U$; mech energy is not conserved in this case]

$$h = \frac{1}{2}mR^2\dot{\theta}^2 \underbrace{\underbrace{- mgR\cos\theta}_{U_{grav}(\theta)} - \frac{1}{2}mR^2\omega^2\sin^2\theta}_{U_{eff}(\theta)}$$

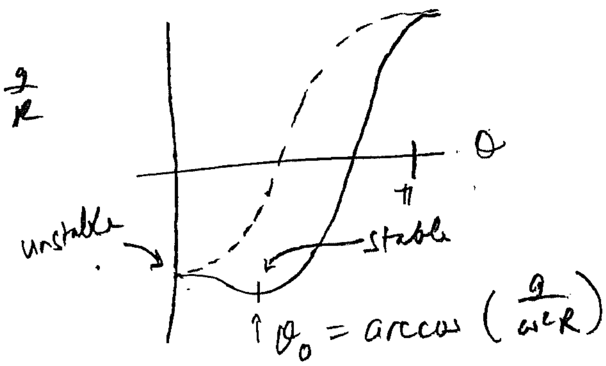


If $\omega^2 < \frac{g}{R}$





If $\omega^2 > \frac{g}{R}$



[Imagine starting with ω small & gradually increasing]

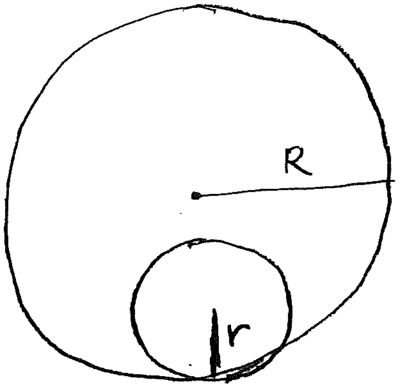
As $\omega \uparrow$, bead moves up the wire

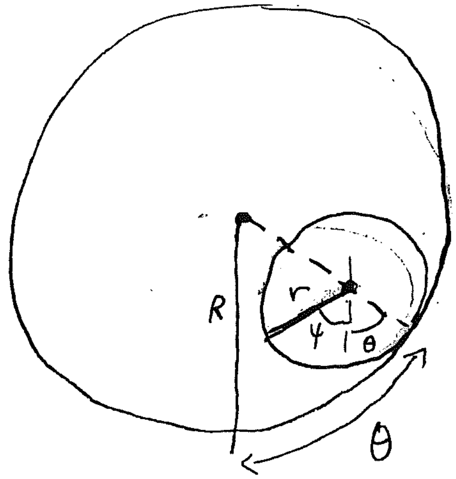
As $\omega \to \infty$, $\theta_0 \to \frac{\pi}{2}$

James Watt (1736-1829) used such a device as a regulator to control speed of steam engine

[cf Taylor p. 264]

(5) Pipe rolling inside a fixed larger pipe (w/o slipping)





ψ = angle inner pipe has turned wrt vertical.

Note $R\theta = r(\psi + \theta) \Rightarrow \psi = \left(\frac{R-r}{r}\right)\theta$

$$T = T_{cm} + T'$$

$$= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I' \dot{\psi}^2$$

I' ← mom of inertia about cm.

$$v_{cm} = (R-r)\dot{\theta}$$

$$I' = mr^2 \Rightarrow \frac{1}{2} I' \dot{\psi}^2 = \frac{1}{2} m (R-r)^2 \dot{\theta}^2$$

$$T = m(R-r)^2 \dot{\theta}^2$$

$$U = -mg(R-r)\cos\theta$$

Euler Lagrange:

$$2m(R-r)^2 \ddot{\theta} + mg(R-r)\sin\theta = 0$$

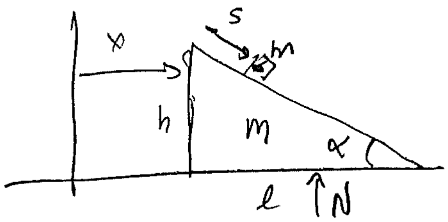
$$\ddot{\theta} + \frac{g}{2(R-r)}\sin\theta = 0$$

↓

or energy approach

(11-22-23)

(solved using Lagrangians Helliwell Sahakian p. 173)



Block on moveable plane solved using energy approach

$$(M+m)x_{cm} = M\left(x+\frac{l}{3}\right) + m(x + s\cos\alpha)$$

$$= M\left(x_0+\frac{l}{3}\right) + m x_0 \qquad (s_0 = 0)$$

$$\Rightarrow M(x-x_0) + m(x-x_0+s\cos\alpha) = 0$$

$$x - x_0 = -\left(\frac{m}{M+m}\right) s\cos\alpha$$

$$(M+m) z_{cm} = M\frac{h}{3} + m(h - s\sin\alpha)$$

$$(M+m)\ddot z_{cm} = -m\ddot s\sin\alpha$$

Now $(M+m)\ddot z_{cm} = F^{ext} = N - (M+m)g$

so can't solve unless know N!

N however does no work so E_{mech} of system is conserved

$$\frac{1}{2}M\dot x^2 + \frac{1}{2}m\left([\dot x + \dot s\cos\alpha]^2 + [\dot s\sin\alpha]^2\right) + mg(h - s\sin\alpha) = mgh$$

$$\underbrace{\frac{1}{2}(M+m)\dot x^2 + m\dot x\underbrace{\dot s\cos\alpha}_{-\left(\frac{M+m}{m}\right)\dot x}}_{-\frac{1}{2}(M+m)\dot x^2 \,=\, -\frac{1}{2}\left(\frac{m^2}{M+m}\right)\dot s^2\cos^2\alpha} + \frac{1}{2}m\dot s^2 = mgs\sin\alpha$$

$$\left[1-\left(\frac{m}{M+m}\right)\cos^2\alpha\right]\dot s^2 = 2g\sin\alpha\, s$$

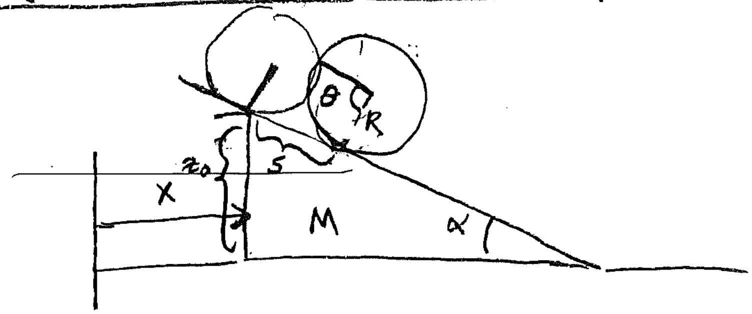
$$\underbrace{\left[1-\left(\frac{m}{M+m}\right)\cos^2\alpha\right]}_{\frac{M-m\sin^2\alpha}{M+m}}\ddot s = g\sin\alpha$$

agrees w/ (4.76)

[cf Taylor, 259]
[Marion, 3e, prob 6.6]

C7

(6) Round object rolling (w/o slipping) down an inclined plane resting on a frictionless surface (2 d.o.f.)



$\theta = 0$ at top of incline

center of mass of round object

$$\begin{cases} x_{cm} = x + s\cos\alpha + R\sin\alpha \\ z_{cm} = z_0 - s\sin\alpha + R\cos\alpha \end{cases}$$

z =

$$s = R\theta$$

$$\Rightarrow \begin{cases} \dot{x}_{cm} = \dot{x} + R\dot{\theta}\cos\alpha \\ \dot{z}_{cm} = -R\dot{\theta}\sin\alpha \end{cases}$$

$$\Rightarrow v_{cm}^2 = \dot{x}^2 + R^2\dot{\theta}^2 + 2R\cos\alpha\,\dot{x}\dot{\theta}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\dot{\theta}^2 - mg\,z_{cm}$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(mR^2+I)\dot{\theta}^2 + mR\cos\alpha\,\dot{x}\dot{\theta} + (mgR\sin\alpha)\,\theta - mgz_0 + \text{const}$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + mR\cos\alpha\,\dot{\theta}$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}} = \text{const}$$

Observe that $\frac{\partial L}{\partial \dot{x}} = p_x^{sys}$

which is conserved because $F_x^{ext} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = (mR^2 + I)\dot{\theta} + mR\cos\alpha\,\dot{x}$$

$$\frac{\partial L}{\partial \theta} = mgR\sin\alpha$$

$$E\text{-}L \Rightarrow (mR^2 + I)\ddot{\theta} + mR\cos\alpha\,\ddot{x} - mgR\sin\alpha = 0$$

Since $p_x = \text{const} \Rightarrow \ddot{x} = -\frac{mR\cos\alpha}{M+m}\ddot{\theta}$

$$\left[(mR^2 + I) - \frac{(mR\cos\alpha)^2}{M+m}\right]\ddot{\theta} = mgR\sin\alpha$$

Since $s = R\theta$, we can write

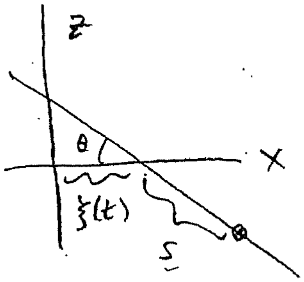
$$\ddot{s} = \frac{g\sin\alpha}{\left[1 + \frac{I}{mR^2} - \frac{m\cos^2\alpha}{M+m}\right]}$$

$\ddot{z} = \ddot{s}\sin\alpha$

[Consider various limits] $M \to \infty \Rightarrow \frac{g\sin\alpha}{1 + \frac{I}{mR^2}}$

$\left(\begin{matrix} m \to \infty \\ I \to 0 \end{matrix} \Rightarrow \frac{g}{\sin\alpha}\right)$

Block on moving incline:



$$x(t) = s\cos\theta + \xi(t)$$
$$z(t) = -s\sin\theta$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{z}^2) - mgz$$
$$= \frac{1}{2}m(\dot{s}^2 + 2\dot{s}\dot{\xi}\cos\theta + \dot{\xi}^2) + mg(\sin\theta)s$$

$$h = \dot{s}\frac{\partial L}{\partial \dot{s}} - L = m\dot{s}^2 + m\dot{\xi}\cos\theta - L$$
$$\boxed{h = \frac{1}{2}m\dot{s}^2 - \frac{1}{2}m\dot{\xi}^2 - mg(\sin\theta)s}$$

If $\frac{\partial L}{\partial t} = 0$, then $\frac{dh}{dt} = 0$, h is conserved

e.g. $\xi = v_0 t \Rightarrow \dot{\xi} = \text{const} \Rightarrow \frac{\partial L}{\partial t} = 0 \Rightarrow \boxed{h = \frac{1}{2}m\dot{s}^2 - mg(\sin\theta)s - \frac{1}{2}mv_0^2}$

$\frac{dh}{dt} = 0$

If $\frac{\partial L}{\partial t} \neq 0$, then h is not conserved, eg $\xi = \frac{1}{2}a_0t^2$

Also if constraints do not depend explicitly on time, $h = T+U = E$

but $\xi = v_0 t$ ← does depend on time so $h \neq E$ $\Rightarrow \boxed{E = \frac{1}{2}m\dot{s}^2 - mg(\sin\theta)s + \frac{1}{2}mv_0^2 + mv_0(\cos\theta)\dot{s}}$

$\frac{dh}{dt} = 0$ while $\frac{dE}{dt} \neq 0$ (plane does work)

If $\xi = \frac{1}{2}a_0t^2$, $h \neq E$ and neither is conserved $\frac{dh}{dt} \neq 0$, $\frac{dE}{dt} \neq 0$