Lagrangian approach to mechanics.

1788

Lagrange (1736 - 1813)

- an elegant alternative to Newtonian approach based on a minimum principle, the principle of least action
- easier because uses scalar rather than vector
- especially useful for systems involving constraints or in non Cartesian coordinates
- essential for modern understanding of QM + QFT

Introduce a Lagrangian function $L(x(t), \dot{x}(t), t)$

$$\dot{x} = \frac{dx}{dt}$$

Consider all paths $x(t)$ ("motions") from $x(t_0) = x_0$ to $x(t_1) = x_1$

Define the action

$$S[x(t)] = \int_{t_0}^{t_1} dt \; L(x(t), \dot{x}(t), t]$$

Principle of least action (Hamilton 1834)

the classical motion of a point particle $x(t)$ between fixed endpoints minimizes (extremizes) the action

7. VARIATIONAL NOTATION

Mary Boas, 2e, p 403–4

The symbol δ was used in the early days of the development of the calculus of variations to indicate what we have called differentiation with respect to the parameter ϵ. It is just like the symbol d in a differential except that it warns you that ϵ and not x is the differentiation variable. The δ notation is not used much any more in mathematics, but you will find it in applications and so should understand its meaning. The quantity δI is just the differential

$$\delta I = \frac{dI}{d\epsilon}\, d\epsilon,$$

where $dI/d\epsilon$ is evaluated for $\epsilon = 0$. The symbol δ (read "the variation of") is also treated as a differential operator acting on F, y, and y'; we shall define δy, $\delta y'$, and δF in terms of our previous notation. We had in Section 2:

$$\begin{aligned} Y(x, \epsilon) &= y(x) + \epsilon\eta(x), \\ Y'(x, \epsilon) &= y'(x) + \epsilon\eta'(x). \end{aligned} \tag{7.1}$$

Then the meaning of δy is

$$\delta y = \left(\frac{\partial Y}{\partial \epsilon}\right)_{\epsilon=0} d\epsilon = \eta(x)\, d\epsilon; \tag{7.2}$$

this is just like a differential dY if ϵ is the variable. The meaning of $\delta y'$ is

$$\delta y' = \left(\frac{\partial Y'}{\partial \epsilon}\right)_{\epsilon=0} d\epsilon = \eta'(x)\, d\epsilon. \tag{7.3}$$

This is identical with

$$\frac{d}{dx}(\delta y) = \frac{d}{dx}[\eta(x)\, d\epsilon] = \eta'(x)\, d\epsilon \tag{7.4}$$

since x and ϵ are independent variables; in other words, d and δ commute. The meaning of δF is

$$\delta F = \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y'; \tag{7.5}$$

this is just a total differential $dF = (\partial F/\partial\epsilon)_{\epsilon=0}\, d\epsilon$ of the function $F[x,\ Y(x, \epsilon),\ Y'(x, \epsilon)]$ at $\epsilon = 0$ with ϵ considered the only variable. Then the variation in I is

$$\begin{aligned} \delta I &= \delta \int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} \delta F\, dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y'\right) dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\,\eta(x)\, d\epsilon + \frac{\partial F}{\partial y'}\,\eta'(x)\, d\epsilon\right] dx. \end{aligned} \tag{7.6}$$

If you compare (7.6) with (2.13), you find that the following two statements about $I = \int F(x, y, y')\, dx$ mean the same thing:

(a) I is stationary; that is, $dI/d\epsilon = 0$ at $\epsilon = 0$ as in (2.13).
(b) The variation of I is zero; that is, $\delta I = 0$ as in (7.6).

[Introduce a shorthand notation]

"δ-notation" [Boas, 2e, p 403-4]

$$S[x] = \int dt \; L(x, \dot{x}, t)$$

Recall family of paths

$$x_\epsilon(t) = \bar{x}(t) + \epsilon\,\eta(t)$$

$$\dot{x}_\epsilon(t) = \dot{\bar{x}}(t) + \epsilon\,\dot{\eta}(t) \qquad \text{dot} = \frac{d}{dt}$$

$$\frac{dS[x_\epsilon(t)]}{d\epsilon} = \int dt \left[\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta} \right]$$

$$\frac{dS}{d\epsilon}\cdot d\epsilon = \int dt \left[\frac{\partial L}{\partial x}\eta\, d\epsilon + \frac{\partial L}{\partial \dot{x}}\dot{\eta}\, d\epsilon \right] \quad (*)$$

Define "variation of x"

$$\left.\begin{aligned} \delta x &\equiv \left.\frac{\partial x_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} d\epsilon = \eta\, d\epsilon \\ \delta \dot{x} &= \left.\frac{\partial \dot{x}_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} d\epsilon = \dot{\eta}\, d\epsilon \end{aligned}\right\} \quad (**)$$

$$\delta S = \left.\frac{\partial S}{\partial \epsilon}\right|_{\epsilon=0} d\epsilon$$

Then (*) is

$$\delta S = \int dt \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right] \qquad \text{"chain rule for } \delta\text{"}$$

Observe that $\delta\dot{x} = \delta\left(\frac{dx}{dt}\right) = \frac{d}{dt}(\delta x)$ from (**)

as δ and $\frac{d}{dt}$ commute

$$\delta S = \int dt \left[\frac{\partial L}{\partial x}\delta x + \underbrace{\frac{\partial L}{\partial \dot{x}} \frac{d}{dt}(\delta x)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x} \right]$$

$$= \int dt \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x + \underbrace{\int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right)}_{\left.\frac{\partial L}{\partial \dot{x}}\delta x\right|_{t_1}^{t_2} = 0}$$

because $\delta x(t_1) = \delta x(t_2) = 0$

$\delta S = 0$ for any variation δx

$$\rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Euler-Lagrange equation (1st form)

If we define $L = T - U$ [units of energy, but not energy]

for a nonrelativistic, conservative system [Goldstein, 1e, 206]

then Euler-Lagrange eqns are equivalent to

Newton's $2^{\underline{nd}}$ law.

Consider Cartesian coordinate x_i for a point particle

$$(x_1, x_2, x_3) = (x, y, z)$$

$$L = \frac{1}{2} m \sum_{j=1}^{3} \dot{x}_j^2 - U(x_i)$$

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i}$$

$$\Rightarrow \text{E-L:} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$m\ddot{x}_i + \frac{\partial U}{\partial x_i} = 0$$

$$m\ddot{x}_i = F_i \qquad \left(F_i = -\frac{\partial U}{\partial x_i}\right)$$

Since L = scalar, can rewrite it in coordinates other than Cartesian, or indeed in terms of any set of

<u>generalized coordinates</u> q_i, $i = 1, \ldots, d$

equal to # of degrees of freedom of the system (complete and independent)

$$S = \int_{t_0}^{t_1} dt \; L(q_i, \dot{q}_i)$$

where

$$\dot{q}_i = \frac{dq_i}{dt} = \text{generalized velocities}$$

Space of generalized coordinates is called <u>configuration space</u>

Principle of stationary action $\Rightarrow$ Euler Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad (i = 1, \ldots, d)$$

Define

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} = \text{generalized momenta (or momentum conjugate to } q_i\text{)}$$

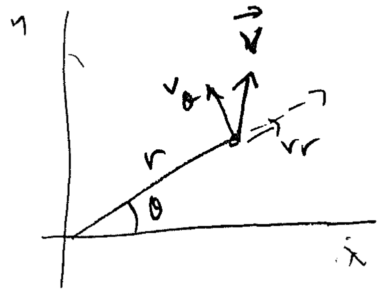
$$F_i \equiv \frac{\partial L}{\partial q_i} = \text{generalized force}$$

$$\Rightarrow \frac{dp_i}{dt} = F_i$$

Corollary: if L is <u>independent</u> of one (or more) generalized coordinates, ie $\frac{\partial L}{\partial q_i} = 0$

q_i is called a "cyclic variable" and its conjugate momentum p_i is conserved.

Point particle in plane using polar coordinates as generalized coordinate (r, θ)



$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$
$$v^2 = \dot{r}^2 + (r\dot{\theta})^2$$

[or can use $x = r\cos\theta$, $y = r\sin\theta$ + chain rule]

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r, \theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0$$

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) + \frac{\partial U}{\partial \theta} = 0$$

$$m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = -\frac{\partial U}{\partial \theta}$$

Now $\vec{F} = -\vec{\nabla} U = \left(-\frac{\partial U}{\partial r}, -\frac{1}{r}\frac{\partial U}{\partial \theta}\right) = (F_r, F_\theta)$

$$m(\ddot{r} - r\dot{\theta}^2) = F_r \quad \Rightarrow \quad a_r = \ddot{r} - r\dot{\theta}^2$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta \quad \Rightarrow \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

rev 2024

1st form of Euler-Lagrange: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \ldots d$

$L(q_i, \dot{q}_i, t)$

Consider

$$\frac{dL}{dt} = \sum_i \Big(\underbrace{\frac{\partial L}{\partial q_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \text{ by 1st form}} \frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \Big) + \frac{\partial L}{\partial t}$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \frac{dq_i}{dt} \right) + \frac{\partial L}{\partial t}$$

$$0 = \frac{d}{dt}\left(\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right) + \frac{\partial L}{\partial t}$$

2nd form of E-L (only one eqn)

Define $h = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$

$$0 = \frac{dh}{dt} + \frac{\partial L}{\partial t}$$

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① If L does not depend explicitly on time $\left(\frac{\partial L}{\partial t} = 0\right)$

then $h = \text{const}$ in time ie h is conserved

② If T depends quadratically on $\dot{q}_i$ and U is independent of $\dot{q}_i$

then h = mechanical energy

$$T = \sum_i \sum_j f_{ij}(q)\, \dot{q}_i \dot{q}_j$$

[usually f_{ij} is diagonal $T = \sum_i f_{ii} \dot{q}_i^2$ for orthogonal coord. systems]

$$\frac{\partial T}{\partial \dot{q}_i} = 2 \sum f_{ij}(q)\, \dot{q}_j$$

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2 \sum f_{ij}\, \dot{q}_i \dot{q}_j = 2T$$

$$h = \sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - L = 2T - (T - U) = T + U = E_{mech}$$

③ If both ① & ② are true, then mechanical energy is conserved

[earlier: E_{mech} conserved if system is isolated, + all forces are conservative & constraint (non dissipative)]

[T quadratic in $\dot{q}$ ⇒ no moving constraints]

(probably 1994 notes)

K4

h conserved if $\frac{\partial L}{\partial t} = 0$,

Does $h = T + V$?

Yes, if ⓐ V is velocity independent [not true if magnetic field]

ⓑ constraints are scleronomic,

i.e. relations between Cartesian + generalized coords do <u>not</u> depend on time

$$x_i = x_i(q, \cancel{t})$$

$$ⓐ \Rightarrow \frac{\partial V}{\partial \dot{q}_i} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

$$ⓑ \Rightarrow T = a_{jk}(q)\, \dot{q}_j \dot{q}_k + \underbrace{b_i \dot{q}_i}_{0} + \underbrace{c}_{0}$$

since $\frac{\partial x_i}{\partial t} = 0$

$$\frac{\partial T}{\partial \dot{q}_i} = a_{jk} \dot{q}_j \underbrace{\frac{\partial \dot{q}_k}{\partial \dot{q}_i}}_{\delta_{ki}} + a_{jk} \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{q}_i}}_{\delta_{ji}} \dot{q}_k$$

$$= a_{ji} \dot{q}_j + a_{ik} \dot{q}_k$$

$$= 2 a_{ij} \dot{q}_j$$

$$\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2 a_{ij} \dot{q}_i \dot{q}_j = 2T$$

$$h = 2T - L = T + V \quad ✓$$