Calculus of variations

[Feynman vol II, ch 19]

Consider a function $f(x)$

What happens to f if we vary independent variable x?

Derivative $\frac{df}{dx}$

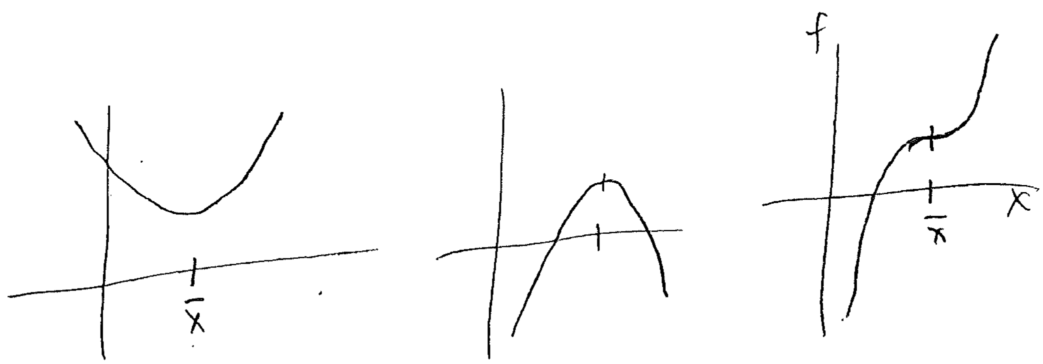
If f doesn't change at $\bar{x}$

$$\left.\frac{df}{dx}\right|_{x=\bar{x}} = 0$$

then $\bar{x}$ is a stationary point of f

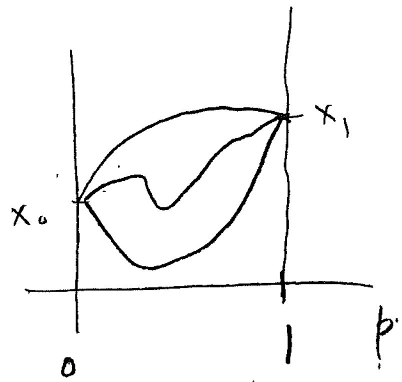
[max/min/pt of inflection]

$\underbrace{\text{max/min}}_{\text{extremum}}$



Now consider a family of continuous functions $x(p)$ ← parameter

w/ fixed endpoints: $\begin{cases} x(0) = x_0 \\ x(1) = x_1 \end{cases}$



paths from x_0 to x_1

Let $J[x(p)]$ be a function of these paths

We call this a "functional" [square brackets]

What happens to J if we vary the path $x(p)$?

~~Functional~~ derivative $\frac{\delta J}{\delta x}$

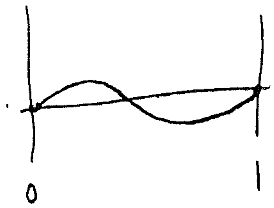
Consider some particular path $\bar{x}(p)$

If J doesn't change for paths near $\bar{x}(p)$

$$\left.\frac{\delta J}{\delta x}\right|_{x(p)=\bar{x}(p)} = 0$$

then $\bar{x}(p)$ is a stationary pt. of J.

Let $\eta(p)$ be any arbitrary function that vanishes at endpoints w/ $\begin{cases} \eta(0) = 0 \\ \eta(1) = 0 \end{cases}$

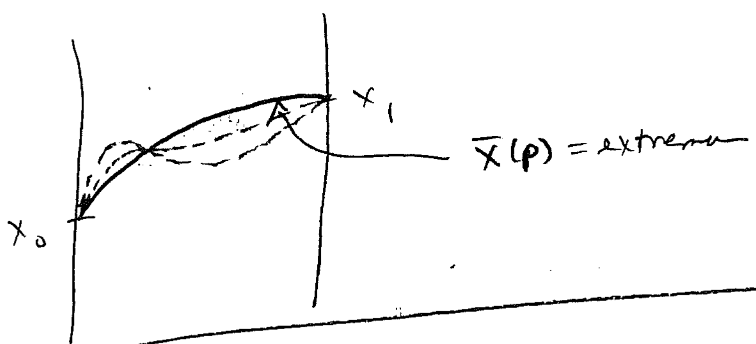


Define a one-parameter family of paths near $\bar{x}(p)$

$$x_\epsilon(p) \doteq \bar{x}(p) + \epsilon\,\eta(p)$$

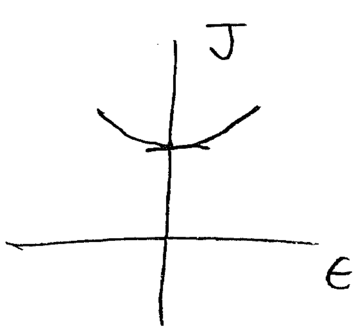
observe $\begin{cases} x_\epsilon(0) = x_0 \\ x_\epsilon(1) = x_1 \end{cases}$

[so all these are indeed paths from x_0 to x_1]



If $\bar{x}(p)$ is a minimum of J, ~~then~~

In particular $J[x_\epsilon(p)] \geq J[\bar{x}(p)]$



$$\Rightarrow \left. \frac{dJ[x_\epsilon(p)]}{d\epsilon} \right|_{\epsilon=0} = 0$$

for *any* choice of $\eta(p)$

More generally, this is true if $\bar{x}(p)$ is a stationary point of J [could be saddle point]

Suppose $J[x(p)]$ is given by the integral

$$\int_0^1 dp \; f(x(p), x'(p), p) \qquad \text{where } x' = \frac{dx}{dp}$$

then

$$J[x_\epsilon(p)] = \int_0^1 dp \; f(\bar{x} + \epsilon\eta, \bar{x}' + \epsilon\eta', p)$$

Use chain rule to take derivative

$$\frac{dJ}{d\epsilon} = \int_0^1 dp \left[\frac{\partial f}{\partial x} \frac{d(\bar{x}+\epsilon\eta)}{d\epsilon} + \frac{\partial f}{\partial x'} \frac{d(\bar{x}'+\epsilon\eta')}{d\epsilon} \right]$$

$$= \int_0^1 dp \left[\frac{\partial f}{\partial x} \eta + \frac{\partial f}{\partial x'} \eta' \right]$$

Recall $\frac{\partial f}{\partial x'} \frac{d\eta}{dp} = \frac{d}{dp}\left(\frac{\partial f}{\partial x'} \eta \right) - \frac{d}{dp}\left(\frac{\partial f}{\partial x'} \right) \eta$

$$\frac{dJ}{d\epsilon} = \int_0^1 dp \left[\frac{\partial f}{\partial x} - \frac{d}{dp}\left(\frac{\partial f}{\partial x'} \right) \right] \eta + \underbrace{\int_0^1 dp \frac{d}{dp}\left(\frac{\partial f}{\partial x'} \eta \right)}_{\underbrace{\left. \frac{\partial f}{\partial x'} \eta(p) \right|_0^1}_{\text{vanishes because } \eta(0) = \eta(1) = 0}}$$

$$\frac{dJ}{d\epsilon} = \int_0^1 dp \left[\frac{\partial f}{\partial x} - \frac{d}{dp}\left(\frac{\partial f}{\partial x'} \right) \right] \eta$$

If $\bar{x}(p)$ is a stationary point of J

then $\frac{dJ}{d\epsilon} = 0$ for any $\eta(p)$.

This can only occur if

$$\boxed{\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) - \frac{\partial f}{\partial x} = 0}$$

Euler's eqn (1744)

Corollary: if f is independent of x
(ie it depends only on x') then

$$\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) = 0$$

ie $\frac{\partial f}{\partial x'}$ = constant (indep of p)

Second form of Euler's eqn

Given $f(x, x', p)$

Consider
$$\frac{df}{dp} = \underbrace{\frac{\partial f}{\partial x}}_{\substack{\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) \\ \text{by Euler (first form)}}} \frac{dx}{dp} + \frac{\partial f}{\partial x'}\frac{dx'}{dp} + \frac{\partial f}{\partial p}$$

$$= \frac{d}{dp}\left(x' \frac{\partial f}{\partial x'}\right) + \frac{\partial f}{\partial p}$$

$$\boxed{0 = \frac{d}{dp}\left(x' \frac{\partial f}{\partial x'} - f\right) + \frac{\partial f}{\partial p}}$$

2nd form of Euler's eqn

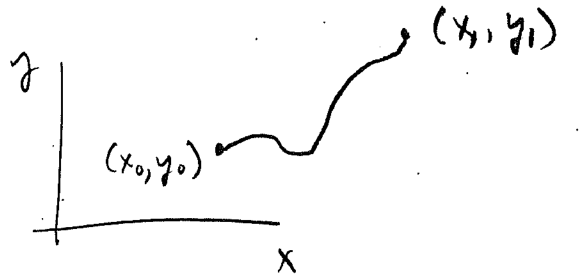
Corollary 2: if f has no explicit dependence on p

ie $\frac{\partial f}{\partial p} = 0$ then

$$\frac{d}{dp}\left(x' \frac{\partial f}{\partial x'} - f\right) = 0$$

ie $x' \frac{\partial f}{\partial x'} - f = \text{const}$ (indep of p)

Shortest distance between 2 pts on a plane is a straight line



Length of an arbitrary path $L = \int ds = \sqrt{dx^2 + dy^2}$

choose x as independent variable $L = \int dx \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}_{f(y, y', x)}$

First form of Euler: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

Since f indep. of y, $\frac{\partial f}{\partial y'} = \text{const}$

$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \Rightarrow y' = \text{const}, C \Rightarrow y = Cx + D$

use endpts to fix constants: $y = y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$

Alternatively, second form of Euler: $\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'} - f\right) - \frac{\partial f}{\partial x} = 0$

Since f indep of x, $\underbrace{y' \frac{\partial f}{\partial y'} - f}_{\underbrace{\frac{y'^2}{\sqrt{1+y'^2}} - \sqrt{1+y'^2}}} = \text{const}$

$\frac{-1}{\sqrt{1+y'^2}} = \text{const} \Rightarrow y' = \text{const}$

(same result as before)

Alt. choose a parameter p

$$L = \int dp \underbrace{\sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2}}_{f}$$

f indep of x or y $\Rightarrow \begin{cases} \dfrac{\partial f}{\partial \dot{x}} = \dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = A \\ \dfrac{\partial f}{\partial \dot{y}} = \dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = B \end{cases}$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{B}{A}$$ $\Rightarrow$ straight line.

2nd form of euler: f independent of p so

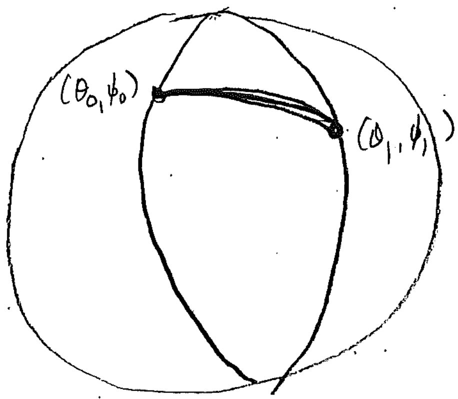
$$\dot{x}\frac{\partial f}{\partial \dot{x}} + \dot{y}\frac{\partial f}{\partial \dot{y}} - f = \frac{\dot{x}^2}{\sqrt{\dot{x}^2+\dot{y}^2}} + \frac{\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}} - \sqrt{\dot{x}^2+\dot{y}^2} = 0$$

so this does not yield anything useful

[Skip b2-4 in 2024]

shortest distance between 2 pts on a sphere is a great circle

cf Marion (3e) p.181



$$L = \int ds = \int \sqrt{(R\,d\theta)^2 + (R\sin\theta\, d\phi)^2}$$

choose ϕ as independent variable

$$L = R \int d\phi \underbrace{\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}}_{f(\theta,\, \theta',\, \phi)}$$

Use 2nd form of Euler:

$$\frac{d}{d\phi}\left(\theta' \frac{\partial f}{\partial \theta'} - f \right) - \frac{\partial f}{\partial \phi} = 0 \qquad \text{where } \theta' = \frac{d\theta}{d\phi}$$

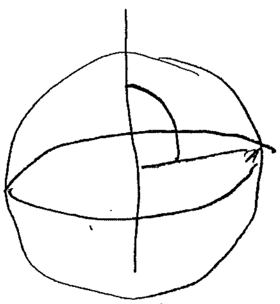
$$\frac{\partial f}{\partial \theta'} = \frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta}} \quad \Rightarrow \quad \theta' \frac{\partial f}{\partial \theta'} - f = \frac{-\sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta}}$$

Since f is indep of ϕ, $\quad \dfrac{\sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta}} = C$

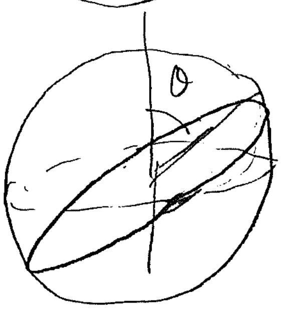
Solve for θ' $\Rightarrow$ $\quad \dfrac{d\theta}{d\phi} = \dfrac{\sin\theta}{C}\sqrt{\sin^2\theta - C^2}$

Before solving this eqn, let's think about what the solution should look like

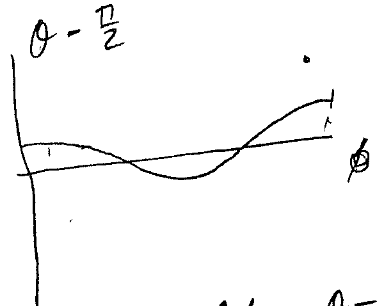
Great circles



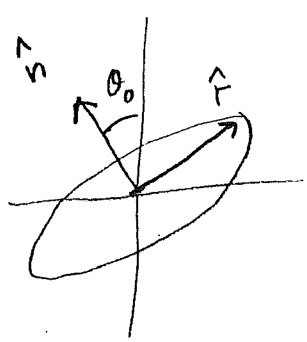
equator $\Rightarrow$ $\theta = \frac{\pi}{2}$



$\Longrightarrow$



Looks approximately like $\theta - \frac{\pi}{2} \sim \cos\phi$



Points on great circle satisfy

$$\hat{r}^2 = 1$$

$\hat{r}\cdot\hat{n} = 0$ for $\hat{n}$ perpendicular to circle

$$\hat{n} = (\cos\theta_0,\ \sin\theta_0\cos\phi_0,\ \sin\theta_0\sin\phi_0)$$

$$\hat{r} = (\cos\theta,\ \sin\theta\cos\phi,\ \sin\theta\cos\phi)$$

$$\hat{r}\cdot\hat{n} = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\,(\underbrace{\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0}_{\cos(\phi-\phi_0)}) = 0$$

$$\Rightarrow \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta_0}{\cos\theta_0}\cos(\phi-\phi_0) = 0$$

ie $\boxed{\cot\theta = -\tan\theta_0\cos(\phi-\phi_0)}$

NB $\cot\theta \sim \frac{\pi}{2} - \theta$ near $\theta = \frac{\pi}{2}$

$$\frac{d\theta}{d\phi} = \frac{\sin\theta}{C}\sqrt{\sin^2\theta - C^2}$$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$$d\phi = \frac{C\,d\theta}{\sin\theta\sqrt{\sin^2\theta - C^2}}$$

$$= \frac{C\,d\theta}{\sin^2\theta\sqrt{1 - C^2\csc^2\theta}}$$

$$= \frac{C\csc^2\theta\,d\theta}{\sqrt{1 - C^2(1+\cot^2\theta)}}$$

$$= \frac{C}{\sqrt{1-C^2}}\frac{\csc^2\theta\,d\theta}{\sqrt{1 - \left(\frac{C^2}{1-C^2}\right)\cot^2\theta}}$$

$$= \frac{A\csc^2\theta\,d\theta}{\sqrt{1-(A\cot\theta)^2}}$$

Let $\xi = A\cot\theta$

$d\xi = -A\csc^2\theta\,d\theta$

$$d\phi = \frac{-d\xi}{\sqrt{1-\xi^2}}$$

$\xi = \cos\chi$

$d\xi = -\sin\chi\,d\chi$

$$d\phi = \frac{\sin\chi\,d\chi}{\sqrt{1-\cos^2\chi}}$$

$$d\phi = d\chi$$

$\Rightarrow \quad \phi = \chi + \phi_0$

$\chi = \phi - \phi_0$

$\xi = \cos\chi = \boxed{\cos(\phi-\phi_0) = A\cot\theta}$ ← great circle if $A = -\cot\theta_0$

Try 1st form of Euler

$$L = \int ds = R \int \sqrt{d\theta^2 + \sin^2\theta \, d\phi^2}$$

$$= R \int d\phi \underbrace{\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}}_{f}$$

$$\frac{\partial f}{\partial \theta} = \frac{\sin\theta\cos\theta}{\sqrt{\dot\theta^2 + \sin^2\theta}}$$

$$\frac{\partial f}{\partial \dot\theta} = \frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta}}$$

$$0 = \frac{d}{d\phi}\left(\frac{\partial f}{\partial \dot\theta}\right) - \frac{\partial f}{\partial \theta}$$

$$= \frac{\ddot\theta}{\sqrt{\quad}} - \frac{\dot\theta(\dot\theta\ddot\theta + \sin\theta\cos\theta\,\dot\theta)}{(\quad)^{3/2}} - \frac{\sin\theta\cos\theta}{\sqrt{\quad}}$$

$$0 = (\dot\theta^2 + \sin^2\theta)\ddot\theta - \dot\theta^2\ddot\theta - \dot\theta^2\sin\theta\cos\theta - (\dot\theta^2 + \sin^2\theta)\sin\theta\cos\theta$$

$$= \sin^2\theta\,\ddot\theta - 2\dot\theta^2\sin\theta\cos\theta - \sin^3\theta\cos\theta$$

$$0 = \ddot\theta - 2\cot\theta\,\dot\theta^2 - \sin\theta\cos\theta$$

$$\dot\theta\ddot\theta - 2\cot\theta\,\dot\theta^3 - \sin\theta\cos\theta\,\dot\theta = 0$$

$$\frac{1}{2}\dot\theta^2 - \qquad -\frac{1}{2}\sin^2\theta$$

[Other approach to great circle not so easy]

$$L = \int ds = R \int dp \underbrace{\sqrt{\left(\frac{d\theta}{dp}\right)^2 + \sin^2\theta \left(\frac{d\phi}{dp}\right)^2}}_{f}$$

2nd form of euler yields zero.
1st form of euler.

$$\frac{\partial f}{\partial \dot\phi} = \frac{\sin^2\theta\ \dot\phi}{\sqrt{\dot\theta^2 + \sin^2\theta\ \dot\phi^2}} = C$$

$$\sin^4\theta\ \dot\phi^2 = c^2(\dot\theta^2 + \sin^2\theta\ \dot\phi^2)$$

$$\dot\theta^2 = \left(\frac{\sin^2\theta}{c^2} - 1\right)\sin^2\theta\ \dot\phi^2$$

$$\frac{\partial f}{\partial \theta} = \frac{\sin\theta\cos\theta\ \dot\phi^2}{\sqrt{\dot\theta^2 + \sin^2\theta\ \dot\phi^2}}$$

$$\frac{\partial f}{\partial \dot\theta} = \frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta\ \dot\phi^2}}$$

$$\frac{d}{dp}\left[\frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta\ \dot\phi^2}}\right] = \frac{\sin\theta\cos\theta\ \dot\phi^2}{\sqrt{\dot\theta^2 + \sin^2\theta\ \dot\phi^2}}$$

Remember we have an arbitrary parameter p here.
We could choose $p = \phi$ in which case $\dot\theta^2$ eqn
reduces to previous