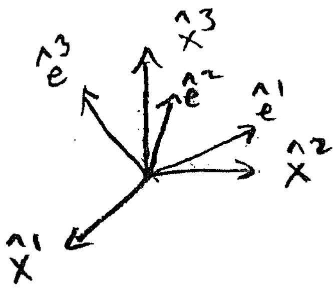
[In 2018, these notes immediately followed U.
In 2024, " " , I think I will defer V, W 'til later in course, after doing Lagrangians, but I might omit it (we lost two classes due to ice storm + eclipse)]

To understand rotational motion in space-fixed frame, need a set of coordinates to describe orientation of object relative to space-fixed axes $\hat{x}^m$

$\Rightarrow$ Euler angles



$$\hat{e}^k = \sum_m R_{km} \hat{x}^m \quad \text{where} \quad R_{km} = e^k_m$$

How many free parameters?

R_{km} has 9.

$$\hat{e}^1 \cdot \hat{e}^1 = \hat{e}^2 \cdot \hat{e}^2 = \hat{e}^3 \cdot \hat{e}^3 = 1$$

$$\hat{e}^1 \cdot \hat{e}^2 = \hat{e}^2 \cdot \hat{e}^3 = \hat{e}^3 \cdot \hat{e}^1 = 0$$

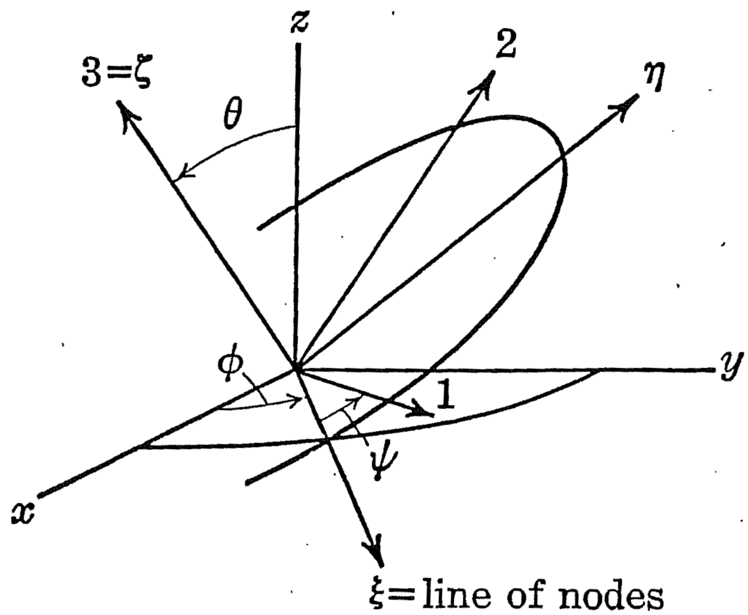
6 conditions

$\Rightarrow$ 3 free parameters

We use Euler angles (ϕ, θ, ψ)

convention of: Slater + Frank
Eck
Goldstein
$\Rightarrow$ Tong

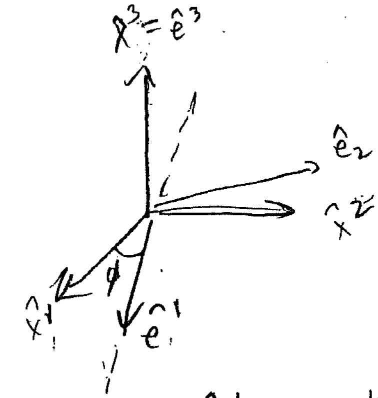
differs from: Taylor
Fetter Walecka


z
$3=\zeta$
2
η
θ
ϕ
y
1
ψ
x
ξ= line of nodes

v2

Start w/ $\hat{x}^m$ and $\hat{e}^k$ aligned.

(1) rotate through ϕ about the $\hat{x}^3 = \hat{e}^3$ axis

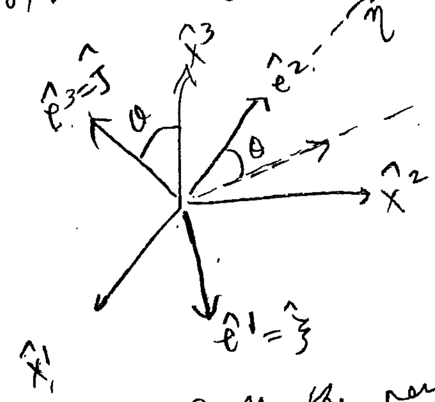


$$R_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_3 \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix} = \begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \\ \hat{e}^3 \end{pmatrix}$$

Call the new $\hat{e}^1$ axis the line of nodes $\hat{\xi}$

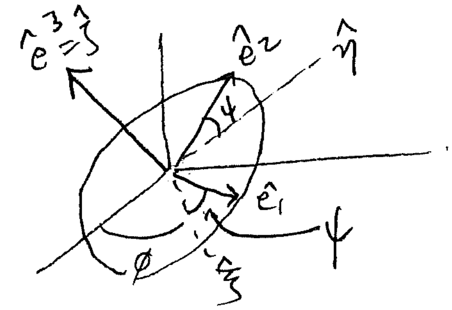
(2) rotate through θ about the line of nodes



$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Call the new $\hat{e}^2$ axis η, and new $\hat{e}^3$ axis ζ

(3) Rotate through ψ about the $\hat{e}^3$ axis



$$R_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[hand out euler angles]

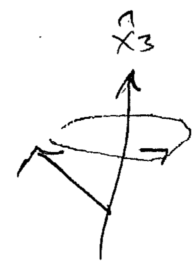
$$R = R_3(\psi) R_1(\theta) R_3(\phi)$$

How do we describe the motions when Euler angles change?

Consider a figure of revolution about $\hat{e}^3$ axis; ($\Rightarrow I_1 = I_2$)

(eg a disk)

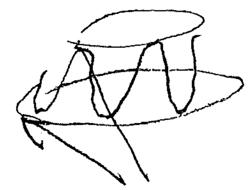
$\dot{\psi} \neq 0 \Rightarrow$ spinning about $\hat{\zeta} = \hat{e}^3$



$\dot{\phi} \neq 0 \Rightarrow \hat{e}^3$ axis circles the $\hat{x}_3$ axis

precession ("wobble")

$\dot{\theta} \neq 0 \Rightarrow$ angle of tip is changing



if $\dot{\theta}$ periodic,

called nutation (nodding)

Let's write $\vec{\omega}$ in terms of Euler angles

From figure — spin rate, precession rate, "tilting" rate

$$\vec{\omega} = \dot{\psi}\,\hat{e}^3 + \dot{\phi}\,\hat{x}_3 + \dot{\theta}\,\hat{\xi}$$

not orthogonal!

$$\hat{x}_3 = \hat{e}^3 \cos\theta + \hat{\eta}\sin\theta$$

$$\vec{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}^3 + \dot{\phi}\sin\theta\,\hat{\eta} + \dot{\theta}\,\hat{\xi}$$

$$\hat{\eta} = \hat{e}^1 \sin\psi + \hat{e}^2 \cos\psi$$
$$\hat{\xi} = -\hat{e}^1 \cos\psi - \hat{e}^2 \sin\psi$$

$$\vec{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}^3 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\,\hat{e}^1$$
$$+ (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\,\hat{e}^2$$

· Slater & Frank
· Tong

$$\boxed{\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}}$$

← NB. ω_k here are components in the body fixed frame (should have primes but we continue to omit them)

Euler angles and angular velocity

```
In[61]:= R1 = {{Cos[ph[t]], Sin[ph[t]], 0}, {-Sin[ph[t]], Cos[ph[t]], 0}, {0, 0, 1}};

In[62]:= R2 = {{1, 0, 0}, {0, Cos[th[t]], Sin[th[t]]}, {0, -Sin[th[t]], Cos[th[t]]}};

In[63]:= R3 = {{Cos[ps[t]], Sin[ps[t]], 0}, {-Sin[ps[t]], Cos[ps[t]], 0}, {0, 0, 1}};

In[64]:= R[t_] := R3.R2.R1

In[65]:= MatrixForm[R[t]]
Out[65]//MatrixForm=
( Cos[ph[t]] Cos[ps[t]] - Cos[th[t]] Sin[ph[t]] Sin[ps[t]]   Cos[ps[t]] Sin[ph[t]] + Cos[
  -Cos[ps[t]] Cos[th[t]] Sin[ph[t]] - Cos[ph[t]] Sin[ps[t]]  Cos[ph[t]] Cos[ps[t]] Cos[th
                     Sin[ph[t]] Sin[th[t]]                                   -Cos[ph[t]]

In[66]:= omega = Simplify[D[R[t], t].Transpose[R[t]]];

In[67]:= omega[[2, 3]]
Out[67]= Sin[ps[t]] Sin[th[t]] ph'[t] + Cos[ps[t]] th'[t]
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$= \omega_1$

```
In[68]:= omega[[3, 1]]
Out[68]= Cos[ps[t]] Sin[th[t]] ph'[t] - Sin[ps[t]] th'[t]
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$= \omega_2$

```
In[69]:= omega[[1, 2]]
Out[69]= Cos[th[t]] ph'[t] + ps'[t]
```

$= \omega_3$

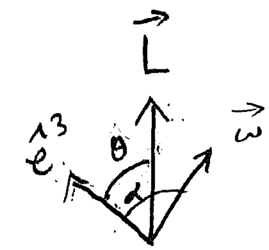
$$(\omega)_{kl} = (\dot{R} R^T)_{kl}$$

$$\omega_{kl} = \epsilon_{kla} \omega_a$$

Revisit motion of free symmetric top, now in the space-fixed frame.

In space fixed (inertial) frame, $\vec{L}$ is conserved, since $\vec{\tau}^{ext} = 0$.

For convenience choose $\vec{L}$ along $\hat{x}^3$ axis



Recall that $\hat{e}^3$, $\vec{\omega}$, $\vec{L}$ are coplanar w/ fixed angles α, θ between them.

N.B. θ is <u>same</u> as Euler angle θ

Since θ is const, $\dot{\theta} = 0$ and so body-fixed components of $\vec{\omega}$ are

$$\left.\begin{aligned} \omega_1 &= \dot{\phi} \sin\theta \sin\psi \\ \omega_2 &= \dot{\phi} \sin\theta \cos\psi \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}\right\} \Rightarrow \omega_1^2 + \omega_2^2 = \dot{\phi}^2 \sin\theta$$

But we know

$$\left.\begin{aligned} \omega_1 &= \omega_\perp \sin\Omega t \\ \omega_2 &= \omega_\perp \cos\Omega t \\ \omega_3 &= \omega_3 = \text{const} \end{aligned}\right\} \Rightarrow \omega_1^2 + \omega_2^2 = \omega_\perp^2 = \text{const}$$

$$\Rightarrow \omega_\perp = \dot{\phi}\sin\theta \quad \Rightarrow \dot{\phi} \text{ is constant}$$

$$\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta \quad \Rightarrow \dot{\psi} \text{ is constant}$$

$$\psi = \Omega t \quad \Rightarrow \dot{\psi} = \Omega = \left(\frac{I_\perp - I_3}{I_\perp}\right)\omega_3$$

From picture $\frac{\omega_\perp}{\omega_3} = \tan\alpha$

Also recall $I_\perp \tan\alpha = I_3 \tan\theta$

Thus $\dot\phi = \frac{\omega_\perp}{\sin\theta} = \frac{\tan\alpha}{\sin\theta}\,\omega_3 = \left(\frac{I_3}{I_\perp \cos\theta}\right)\omega_3$ (precession rate)

Finally, check $\dot\psi = \omega_3 - \dot\phi\cos\theta$

$$= \left(1 - \frac{I_3}{I_\perp}\right)\omega_3$$

$$= \left(\frac{I_\perp - I_3}{I_\perp}\right)\omega_3$$ agrees w/ previous (spin rate)

[makes sense: Ω is rotation rate of $\vec\omega$ & $\vec L$ about body fixed axes
If $\dot\psi = 0 \Rightarrow$ pure precession $\Rightarrow \vec L$ & $\vec\omega$ remain fixed in body-fixed frame]