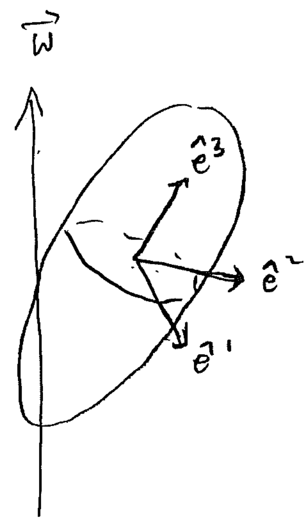


Suppose a rigid body rotates about axis $\vec{\omega}$ at time t

We call $\vec{\omega}$ the <u>instantaneous</u> axis of rotation because it can change (both magnitude + direction) over time

Earlier, we saw that a point $\vec{r}$ in object has velocity

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r}$$

Any vector <u>fixed</u> in the body frame changes as

[cf Taylor 9.23]

$$\frac{d\hat{e}^k}{dt} = \vec{\omega} \times \hat{e}^k$$

my idea!

[Imagine two points fixed in solid separated by $\hat{e}^k$, i.e. $\vec{r}_1, \vec{r}_2 = \vec{r}_1 + \hat{e}^k$

$$\left(\frac{d\hat{e}^k}{dt} = \frac{d}{dt}(\vec{r}_2 - \vec{r}_1) = \vec{\omega} \times \vec{r}_2 - \vec{\omega} \times \vec{r}_1 \right.$$
$$= \vec{\omega} \times (\vec{r}_2 - \vec{r}_1)$$
$$= \vec{\omega} \times \hat{e}^k \Big]$$

Consider an arbitrary vector $\vec{A}$
(not necessarily fixed in the body frame)

$$\vec{A} = \sum A_k' \hat{e}^k$$

$$\frac{d\vec{A}}{dt} = \sum \frac{dA_k'}{dt}\hat{e}^k + \underbrace{\sum_k A_k' \underbrace{\frac{d\hat{e}^k}{dt}}_{\vec{\omega}\times\hat{e}^k}}_{\vec{\omega}\times\underbrace{\sum A_k'\hat{e}^k}_{\vec{A}}}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{space} = \left(\frac{d\vec{A}}{dt}\right)_{body} + \vec{\omega}\times\vec{A}$$

↑ change of $\vec{A}$ in space-fixed frame

↑ change w.r.t. body-fixed frame

Earlier we proved $\dfrac{d\vec{L}^{\,sys}}{dt} = \vec{\tau}^{\,ext}$

in any IRF (or in the cm frame even if not inertial)

Consider a rigid object subject to <u>no</u> <u>external</u> <u>torques</u>

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = 0$$ because space-fixed frame is inertial

$\Rightarrow \vec{L} = \text{const}$ (conserved)

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = 0$$

$\vec{L}$ is not constant in body-fixed frame (rotating, not inertial)

Express eqn in body-fixed coordinates.

$$\vec{\omega} = \sum \omega'_k \hat{e}^k$$

$$\vec{L} = \sum L'_k \hat{e}^k = \sum I_k \omega'_k \hat{e}^k$$

$$\left(\frac{d\vec{L}}{dt}\right)_{body} = \sum \frac{dL'_k}{dt} \hat{e}^k = \sum I_k \frac{d\omega'_k}{dt} \hat{e}^k$$

constant in body frame

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{e}^1 & \hat{e}^2 & \hat{e}^3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ I_1\omega'_1 & I_2\omega'_2 & I_3\omega'_3 \end{vmatrix} = \hat{e}^1 \omega_2 \omega_3 (I_3 - I_2) + \ldots$$

Set coefficients equal

$$\boxed{\begin{aligned} I_1 \frac{d\omega'_1}{dt} + (I_3 - I_2)\,\omega'_2 \omega'_3 &= 0 \\ I_2 \frac{d\omega'_2}{dt} + (I_1 - I_3)\,\omega'_1 \omega'_3 &= 0 \\ I_3 \frac{d\omega'_3}{dt} + (I_2 - I_1)\,\omega'_1 \omega'_2 &= 0 \end{aligned}}$$

Euler's equations for rigid body in body-fixed frame

Non linear eqns in $\omega_1, \omega_2, \omega_3$.
Exact solution possible but complicated.

First, assume object is initially rotating with angular speed ω about one of the principal axes (say 3)

$$\omega_1' = \omega_2' = 0, \qquad \omega_3' = \omega$$

Then Euler's eqns $\Rightarrow$ $I_k \dfrac{d\omega_k'}{dt} = 0 \qquad (k = 1, 2, 3)$

$\omega_k' = \text{const}$, ie $\vec{\omega}$ does not change in body fixed frame

$$\left(\frac{d\vec{\omega}}{dt}\right)_{\text{space}} = \underbrace{\left(\frac{d\vec{\omega}}{dt}\right)_{\text{body}}}_{0} + \underbrace{\vec{\omega} \times \vec{\omega}}_{0} = 0$$

$\vec{\omega}$ also does not change in space-fixed frame

$$\left[\; L_k' = I_k \omega_k' = \text{const}, \quad \vec{L} = I_3 \vec{\omega}\right.$$

$$\left.\left(\frac{d\vec{L}}{dt}\right)_{\text{body}} = 0, \qquad \left(\frac{d\vec{L}}{dt}\right)_{\text{space}} = 0 \,.\;\right]$$

Also, this holds if $I_1 = I_2 = I_3 \Rightarrow \omega_k = \text{const}$

<u>all</u> axes are principle axes

Is this motion stable?

What if $\vec{\omega}$ is <u>not</u> exactly along a principal axis?

Suppose $\vec{\omega}$ is initially <u>almost</u> along $\hat{e}^3$

$$\omega_3 \approx \omega$$

$$\omega_1, \omega_2 \ll \omega$$

[We now <u>omit primes</u> for convenience but recall we are in body fixed frame]

Euler: $$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1 \omega_2 \approx 0 \quad \Rightarrow \quad \omega_3 \approx \text{const}$$

$$\frac{d\omega_1}{dt} = \left[\frac{I_2 - I_3}{I_1} \omega_3 \right] \omega_2 \approx \alpha \omega_2 \qquad \text{for const } \alpha$$

$$\frac{d\omega_2}{dt} = \left[\frac{I_3 - I_1}{I_2} \omega_3 \right] \omega_1 \approx \beta \omega_1 \qquad \text{for const } \beta$$

$$\frac{d^2\omega_1}{dt^2} = \alpha \frac{d\omega_2}{dt} = \alpha\beta\, \omega_1$$

$$\frac{d^2\omega_2}{dt^2} = \beta \frac{d\omega_1}{dt} = \beta\alpha\, \omega_2$$

If $\alpha\beta < 0$ $\Rightarrow$ harmonic oscillator $\Rightarrow$ ω_1, ω_2 remain small
Rotation about $\hat{e}^3$ is stable

If $\alpha\beta > 0$ $\Rightarrow$ exponential growth $\Rightarrow$ ω_1, ω_2 get large
Rotation about $\hat{e}^3$ unstable

If $\alpha\beta = 0$ $\Rightarrow$ linear growth $\Rightarrow$ ω_1, ω_2 get large
Rotation unstable

$$\alpha\beta = \left(\frac{I_2 - I_3}{I_1}\omega\right)\left(\frac{I_3 - I_1}{I_2}\omega\right)$$

$$= -\frac{(I_2 - I_3)(I_1 - I_3)}{I_1 I_2}\omega^2$$

If $I_3 < I_1, I_2$ or $I_3 > I_1, I_2$ then $\alpha\beta < 0$

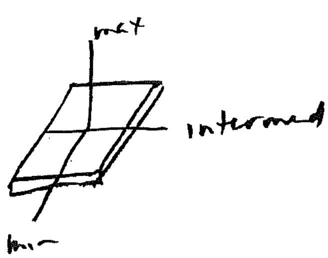
$\Rightarrow$ stable rotation about $\hat{e}^3$ if I_3 is either min or max

If $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$ then $\alpha\beta > 0$

$\Rightarrow$ unstable rotation about $\hat{e}^3$ if I_3 is intermediate between I_1 + I_2

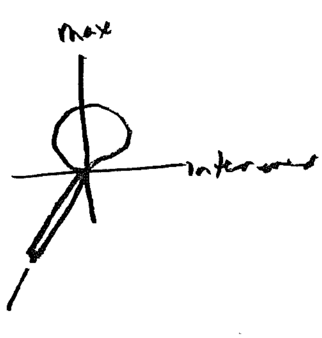
(intermediate axis theorem)

book



Demo: hard to catch when spun about intermediate axis

tennis racket



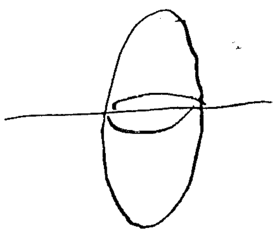
Demo

A symmetric top is an object w/ two equal principle moments

$$I_1 = I_2 = I_\perp \quad \Rightarrow \quad I = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

(body fixed frame)

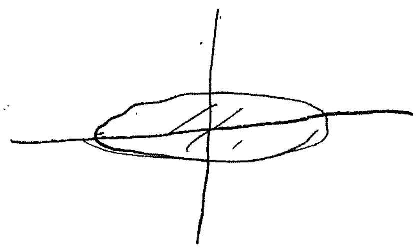
eg a figure of revolution



football (American)

"prolate ellipsoid"

$I_3 < I_\perp$



plate

"oblate ellipsoid"

$I_3 > I_\perp$

$$I_3 \frac{d\omega_3}{dt} + \overbrace{(I_\perp - I_\perp)}^{0}\omega_1\omega_2 = 0 \quad \Rightarrow \quad \omega_3 = \text{const}$$

$$\alpha\beta = \left(\frac{I_\perp - I_3}{I_\perp}\omega_3\right)\left(\frac{I_3 - I_\perp}{I_\perp}\omega_3\right) = -\frac{(I_3 - I_\perp)^2}{I_\perp^2}\omega_3^2 < 0$$

so rotation approximately about $\hat{e}^3$ is stable

[Rotation about $\hat{e}^\perp \Rightarrow \alpha\beta = 0 \Rightarrow$ unstable
[grows linearly w/ time]]

Motion of free symmetric top (free = no torque)

[convention of <u>Taylor</u> <u>Tong</u> <u>Goldstein</u> Eck
opposite sign used in Marion-Thornton Fetter Walecka]

Euler eqns
$$\begin{cases} \dfrac{d\omega_3}{dt} = 0 \Rightarrow \omega_3 \text{ const} \\ \dfrac{d\omega_1}{dt} = \Omega\, \omega_2 \quad \text{where } \Omega \equiv \underbrace{\left(\dfrac{I_\perp - I_3}{I_\perp}\right)\omega_3}_{\text{const}} \\ \dfrac{d\omega_2}{dt} = -\Omega\, \omega_1 \end{cases}$$

exact, no approximations

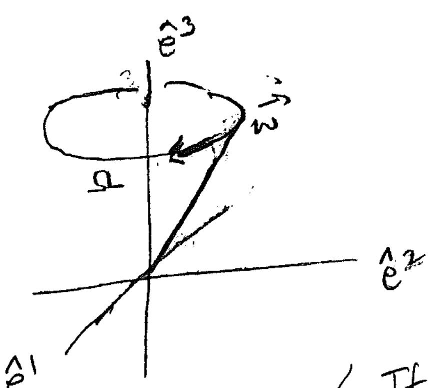
$$\Rightarrow \frac{d^2\omega_1}{dt^2} = -\Omega^2 \omega_1$$

Solution: $\omega_1(t) = \omega_\perp \sin(\Omega t + \varphi)$ $\qquad \omega_\perp, \varphi$ arbitrary determined by initial conditions

$$\Rightarrow \omega_2(t) = \frac{1}{\Omega}\frac{d\omega_1}{dt} = \omega_\perp \cos(\Omega t + \varphi)$$

Let $\varphi = 0$

$$\vec{\omega} = \begin{pmatrix} \omega_\perp \sin \Omega t \\ \omega_\perp \cos \Omega t \\ \omega_3 \end{pmatrix} \quad \text{in body-fixed frame}$$



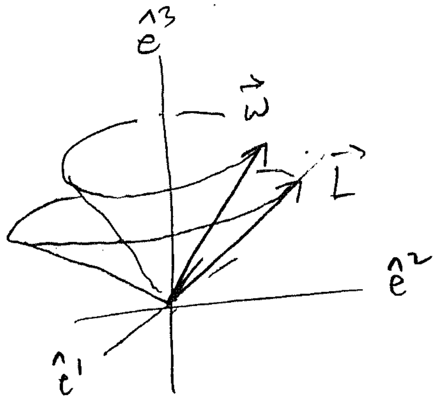
$\vec{\omega}$ moves in a cone about $\hat{e}^3$ w/ angular freq. Ω "body cone"

symmetric top: If $\vec{\omega}$ initially aligned w/ one of principle axes then $\vec{\omega}$ is constant, eg $\omega_3 = 0 \Rightarrow \Omega = 0$ or $\omega_\perp = 0$
but otherwise $\vec{\omega}$ precesses about I_3

$L_k = I_k \omega_k$

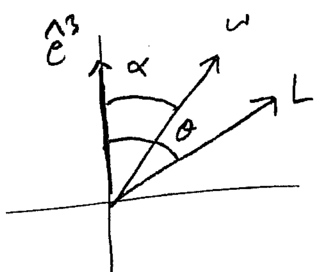
$$\vec{L} = \begin{pmatrix} I_\perp \omega_\perp \sin(\Omega t) \\ I_\perp \omega_\perp \cos(\Omega t) \\ I_3 \omega_3 \end{pmatrix}$$

figure assumes $I_3 < I_\perp$ (prolate)



$\vec{L}$ also moves in a cone about $\hat{e}^3$ w/ angular freq Ω

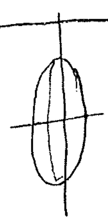
$\hat{e}^3, \vec{\omega}, \vec{L}$ are coplanar



$$\tan\alpha = \frac{\omega_\perp}{\omega_3}$$

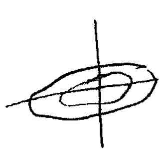
$$\tan\theta = \frac{L_\perp}{L_3} = \frac{I_\perp \omega_\perp}{I_3 \omega_3} = \frac{I_\perp}{I_3}\tan\alpha$$

$$I_\perp \tan\alpha = I_3 \tan\theta$$

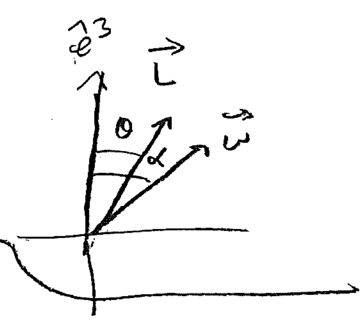


If $I_3 < I_\perp$ (prolate), then $\theta > \alpha$ (as shown)

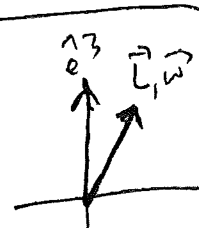
football



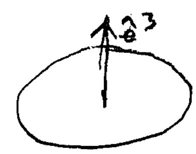
If $I_3 > I_\perp$ (oblate), then $\theta < \alpha$



If $I_3 = I_\perp$ (sphere), then $\Omega = 0$



The earth is not a perfect sphere but an oblate ellipsoid (flattened), so $I_3 > I_\perp$



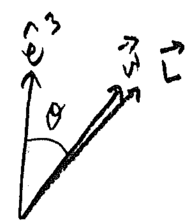
$$\frac{I_3 - I_\perp}{I_\perp} \approx \frac{1}{300}$$

If axis of rotation is not precisely aligned w/ $\hat{e}^3$ it will precess about $\hat{e}^3$ w/ $\Omega \approx -\frac{1}{300}\,\omega_3$

i.e. w/ a period of $\approx$ 300 days

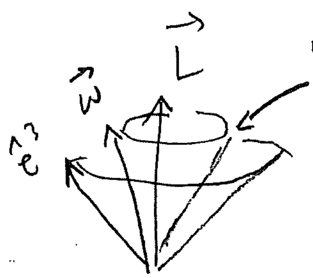
predicted by Euler in 1749
not detected until 1891 by Chandler
"Chandler wobble"
period $\approx$ 435 days [earth not perfectly rigid]

$\vec{\omega}$ intercepts earth's surface ~ 10 m from NP



$$\theta \approx \frac{10\,\text{m}}{6400\,\text{km}} \approx 1.5\times10^{-6}\ \text{rad}$$
$$\approx 0.04\ \text{arcsec}$$

In space-fixed (inertial reference) frame
in which external torque vanishes, $\vec{L}$ is constant!



$\vec{\omega}$ + $\hat{e}^3$ rotate around $\vec{L}$

(but not at frequency Ω)

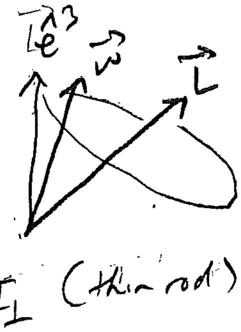
We'll return to this later, after introduce Lagrangian approach & Euler angle

[e.g. if $I_3 = I_\perp$, $\Omega = 0$ but $\hat{e}^3$ "is" spinning around $\vec{\omega} = \vec{L}$]

Stability

If an isolated object (w/ $I_1 = I_2 = I_\perp$) is spinning w/ $\vec{\omega} \parallel \hat{e}^3$
then $\vec{L} \parallel \hat{e}^3$. Then $\vec{L} = \text{const} \Rightarrow \hat{e}^3 = \text{const}$
and object remains stable.

If $\vec{\omega}$ is not quite parallel to $\hat{e}^3$ then body axis will rotate around $\vec{L}$



$I_3 < I_\perp$ (thin rod)
$\Rightarrow \vec{L}$ further away from $\hat{e}^3$
(less stable)

$I_3 > I_\perp$ (disk)
$\Rightarrow \vec{L}$ between $\hat{e}^3$ + $\vec{\omega}$
(more stable)