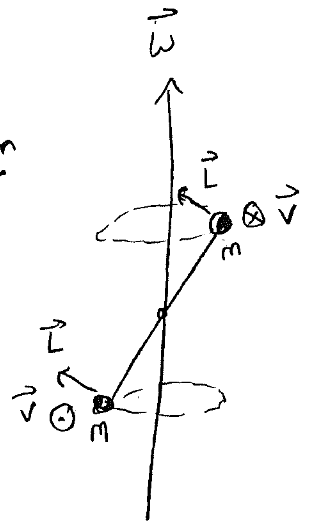
Unbalanced rigid body $\Rightarrow$ $\vec{L}$ not parallel to $\vec{\omega}$

Baton



$$\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i$$

Recall $\vec{v} = \vec{\omega} \times \vec{r}$

[use $\hat{x}, \hat{y}, \hat{z}$ not $\hat{i}, \hat{j}, \hat{k}$ so as not to confuse my indices]

Let $\vec{\omega} = \omega_z \hat{z} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix}$$

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ -\omega y & \omega x & 0 \end{vmatrix} = \begin{pmatrix} -\omega x z \\ -\omega y z \\ \omega x^2 + \omega y^2 \end{pmatrix}$$

$$\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i = \begin{pmatrix} \sum (-m_i x_i z_i) \\ \sum (-m_i y_i z_i) \\ \sum m_i (x_i^2 + y_i^2) \end{pmatrix} \omega \equiv \begin{pmatrix} I_{xz} \\ I_{yz} \\ I_{zz} \end{pmatrix} \omega$$

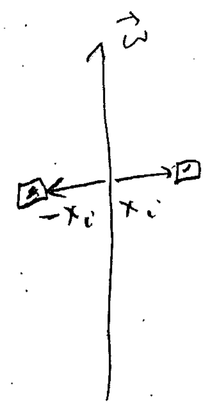
I_{xz}, I_{yz} = products of inertia

I_{zz} = moment of inertia

$$L_x = I_{xz}\,\omega$$
$$L_y = I_{yz}\,\omega$$
$$L_z = I_{zz}\,\omega$$

$\vec{L} \parallel \vec{\omega}$ iff products of inertia vanish

If object is balanced wrt z-axis



$$\sum m_i x_i z_i = 0 \quad \Rightarrow \quad I_{xz} = 0$$

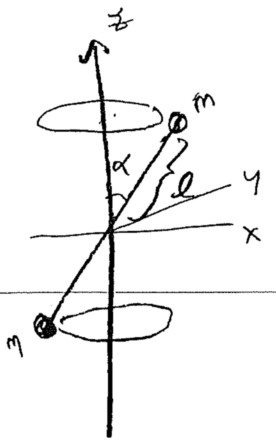
$$\sum m_i y_i z_i = 0 \quad \Rightarrow \quad I_{yz} = 0$$

$$\Rightarrow \vec{L} = I_{zz}\,\hat{z}\,\omega = I_{zz}\,\vec{\omega}$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2) = \sum m_i r_{\perp i}^2$$

as before

Q3



Assume $\vec{\omega} = \omega \hat{z}$

with $\omega = \text{const}$

$$\begin{cases} x = l \sin\alpha \cos\omega t \\ y = l \sin\alpha \sin\omega t \\ z = l \cos\alpha \end{cases}$$

$$I_{xz} = -2ml^2 \sin\alpha \cos\alpha \cos\omega t$$

$$I_{yz} = -2ml^2 \sin\alpha \cos\alpha \sin\omega t$$

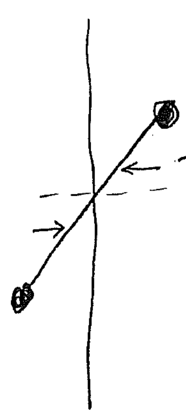
$$I_{zz} = 2ml^2 \sin^2\alpha$$

N.B. $I_{..}$ depend on time (in inertial ref frame)

$$\vec{L} = \begin{pmatrix} I_{xz} \\ I_{yz} \\ I_{zz} \end{pmatrix} \omega = 2ml^2\omega \begin{pmatrix} -\cos\alpha \cos\omega t \\ -\cos\alpha \sin\omega t \\ \sin\alpha \end{pmatrix}$$

$\vec{L}$ not conserved $\Rightarrow$ torque required to maintain rotation

$$\vec{T} = \frac{d\vec{L}}{dt} = 2ml^2\omega \begin{pmatrix} \cos\alpha \sin\omega t \\ -\cos\alpha \cos\omega t \\ 0 \end{pmatrix}$$



These forces supply the necessary torque

[check $\vec{r} \times \vec{F}$ out of page]

"fighting centrifugal force"

[unbalanced washing machine]

Rotation about an arbitrary axis: $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

For an arbitrary mass distribution

$$\vec{L} = \int dm \; \vec{r} \times \vec{v}$$

$$= \int dm \; \vec{r} \times (\vec{\omega} \times \vec{r})$$

Identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

outer dot remote times adjacent

minus outer dot adjacent times remote

$$\vec{L} = \int dm \left[r^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r} \right]$$

$$= \int dm \left[(x^2+y^2+z^2) \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} - (x\omega_x + y\omega_y + z\omega_z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]$$

$$= \int dm \begin{pmatrix} (y^2+z^2)\omega_x - xy\omega_y - xz\omega_z \\ -xy\omega_x + (x^2+z^2)\omega_z - yz\omega_z \\ -xz\omega_x - yz\omega_y + (x^2+y^2)\omega_z \end{pmatrix}$$

$$= \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Note that matrix is symmetric $I^T = I$

Redo using index notation : V_m $(m=1,2,3)$ are Cartesian components of $\vec{V}$

$$L_m = \int dm \, [r^2 \omega_m - (\vec{r}\cdot\vec{\omega}) r_m]$$

$$= \int dm \, [\omega_m \sum r_\ell^2 - r_m \sum r_n \omega_n]$$

$$= \int dm \sum_n [\delta_{mn} \sum r_\ell^2 - r_m r_n] \, \omega_n$$

Recall δ_{mn} = Kronecker delta $= \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

$\sum \delta_{mn} \omega_n = \omega_m$

$$L_m = \sum_n I_{mn} \omega_n$$

where $I_{mn} = \int dm \, [\delta_{mn} \left(\sum r_\ell^2\right) - r_m r_n]$

(Note that $I_{mn} = I_{nm}$ $\Rightarrow$ symmetric matrix) $\rightarrow$ also real $\Rightarrow$ hermitian

Diagonal elements $I_{mm} = \int dm \, [(\sum r_\ell^2) - r_m^2] = \int dm \, r_{\perp m}^2$ = moments of inertia

off-diagonal elements $I_{mn} = - \int dm \, r_m r_n$ = products of inertia

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Another notation

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

"dyad" acts on vector to produce a vector in a (possibly) different direction

Q6

↑ volume of parallelepiped

Kinetic energy

$$T = \int \frac{1}{2}\, dm\, v^2$$

$$= \frac{1}{2}\int dm\ \vec{v}\cdot(\vec{\omega}\times\vec{r})$$

$$= \frac{1}{2}\int dm\ \vec{\omega}\cdot(\vec{r}\times\vec{v})$$

$$= \frac{1}{2}\vec{\omega}\cdot\int dm\,(\vec{r}\times\vec{v})$$

$$= \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

$$= \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$T = \frac{1}{2}\sum_{mn} \omega_m I_{mn}\,\omega_n$$

Identity

$\vec{a}\cdot(\vec{b}\times\vec{c})$

$= \vec{b}\cdot(\vec{c}\times\vec{a})$

$= \vec{c}\cdot(\vec{a}\times\vec{b})$

If rotating around z-axis, reduces to

$$T = \frac{1}{2} I_{zz}\,\omega^2 = \frac{1}{2}\left(\int dm\,(r_{\perp z}^2\right)\omega^2$$

[Talked about Eijk
Levi Civita
$\vec{a} = \vec{b}\times\vec{c}$
$a_i = \sum \epsilon_{ijk}\, b_j c_k$]

Principal axes

R1

If I is a diagonal matrix, i.e. $\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

the object is balanced w.r.t. $\hat{x}, \hat{y}, \hat{z}$ axes

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If $I_{xy} \neq 0$ object is not balanced w.r.t. either $\hat{x}$ + $\hat{y}$ axes

and similarly for I_{xz} + I_{yz}

However, any object, however misshapen, is always

balanced wrt at least 3 axes $\hat{e}^1, \hat{e}^2, \hat{e}^3$

called principal axes

(superscripts label 3 different unit vectors) [subscripts = components]

The moments of inertia w.r.t. these axes, I_1, I_2, I_3

are called principal moments.

Thus if

$\vec{\omega} = \omega \hat{e}^k$ for $k = 1, 2, or\ 3$, then $\vec{L} = I_k \vec{\omega}$

Furthermore, the principal axes are mutually perpendicular

K2

How do we find these principal axes and moments?

In general $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

If $\vec{\omega}$ is parallel to a principal axis $\hat{e}$ then

$$\vec{L} = I\vec{\omega}$$

Thus $\overleftrightarrow{I} \cdot \vec{\omega} = I\vec{\omega}$ in components $\sum_n I_{mn} e_n = I e_m$

or $\overleftrightarrow{I} \cdot \hat{e} = I\hat{e}$

Eigenvalue eqn w/ eigenvalue I and eigenvector $\hat{e}$.

Write it as

$$\left[\overleftrightarrow{I} - I\overleftrightarrow{\mathbb{1}}\right] \cdot \hat{e} = 0$$

where $\overleftrightarrow{\mathbb{1}}$ = identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

or

$$\begin{pmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{pmatrix} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} = 0$$

This has nontrivial solutions for $\hat{e}$ if

$$\underbrace{\det\left(\overleftrightarrow{I} - I\overleftrightarrow{\mathbb{1}}\right)} = 0$$

This is a polynomial cubic in I

Solve the cubic eqn for 3 values of I;

call them I_1, I_2, I_3

(In principle, some of the roots could be complex but for a symmetric matrix, they are all real.)

For each eigenvalue I_k, solve for the corresponding eigenvector $\hat{e}^k$

Eigenvalue equation only determines the direction, not magnitude, of $\hat{e}^k$, so we normalize it to have length 1.

[$\hat{e}^k$ are uniquely determined only if eigenvalues are non degenerate]

$$\sum_n I_{mn} \hat{e}^k_n = I_k e^k_m$$

RY

Proof that eigenvectors are mutually orthogonal.

Let $\hat{e}^k$ and $\hat{e}^l$ be eigenvectors w/ distinct eigenvalues $(I_k \neq I_l)$

$$\overleftrightarrow{I}\cdot\hat{e}^k = I_k\,\hat{e}^k \qquad\qquad \overleftrightarrow{I}\,\hat{e}^l = I_l\,\hat{e}^l$$

Dot w/ $\hat{e}^l$ Dot w/ $\hat{e}^k$

$$(1)\quad \hat{e}^l\cdot\overleftrightarrow{I}\cdot\hat{e}^k = I_k\,\hat{e}^l\cdot\hat{e}^k \qquad (2)\quad \hat{e}^k\cdot\overleftrightarrow{I}\cdot\hat{e}^l = I_l\,\hat{e}^k\cdot\hat{e}^l$$

Claim: $\hat{e}^l\cdot\overleftrightarrow{I}\cdot\hat{e}^k = \hat{e}^k\cdot\overleftrightarrow{I}\cdot\hat{e}^l$ (3)

proof: $\sum_{m,n} e^l_m I_{mn} e^k_n = \sum_{m,n} e^l_m I_{nm} e^k_n = \sum_{m,n} e^k_n I_{nm} e^l_m$

I is symmetric rearrange

$= \sum e^k_m I_{mn} e^l_n = \hat{e}^k\cdot\overleftrightarrow{I}\cdot\hat{e}^l$ QED

relabel $m \leftrightarrow n$

Putting (1), (2), (3) together

$$I_k\,\hat{e}^l\cdot\hat{e}^k = I_l\,\hat{e}^k\cdot\hat{e}^l$$

$$\Rightarrow (I_k - I_l)\,\hat{e}^l\cdot\hat{e}^k = 0$$

But $I_k \neq I_l$ so $\hat{e}^l\cdot\hat{e}^k = 0$ QED omit this unless someone questions it

R5

Proof of reality of eigenvalue

$$\overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^k$$

Dot by $(\hat{e}^k)^*$

$$\hat{e}^{*k} \cdot \overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^{*k} \cdot e^k \quad (1)$$

Take c.c. and use fact that $\overleftrightarrow{I}$ is real

$$\hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^{k*} = I_k^* \hat{e}^k \cdot \hat{e}^{k*} \quad (2)$$

But by previous calc: $\hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^{k*} = \hat{e}^{k*} \cdot \overleftrightarrow{I} \cdot \hat{e}^k \quad (3)$

Putting (1), (2), (3) together

$$I_k \hat{e}^{k*} \cdot e^k = I_k^* \hat{e}^{k*} \cdot e^k$$

$$(I_k - I_k^*) \hat{e}^{k*} \cdot e^k = 0$$

But $\hat{e}^{*k} \cdot \hat{e}^k = \sum |e_i^k|^2 > 0$

so $$I_k - I_k^* = 0$$

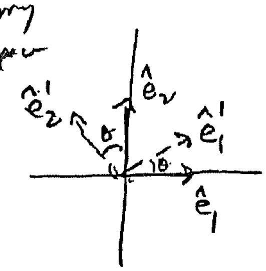
$$I_k^* = I_k \qquad \text{QED}$$

Note, we should also show reality of eigenvectors (i.e. that all the components can be chosen real)

only necessary if I were hermitian but not real

Rotations

N.B. R here is the transpose of my adv. mech. notes, if $\hat{e}_n = \hat{x}^n$ 2 space and $\hat{e}'_k = \hat{e}^k$ body



Let R rotate basis $\hat{e}_n$ into $\hat{e}'_n$

$\hat{e}^k = R\hat{x}^k$

$\hat{e}'_n = R\hat{e}_n$ $\quad = R_{mk} = \hat{x}^m \cdot \hat{e}^k = e^k{}_m \equiv R^{3120}_{km} =$

Define $R_{mn} \equiv \hat{e}_m \cdot R\hat{e}_n = \hat{e}_m \cdot \hat{e}'_n$

$R_{21} = \hat{e}_2 \cdot \hat{e}'_1 = \sin\theta$

$R_{12} = \hat{e}_1 \cdot \hat{e}'_2 = -\sin\theta$

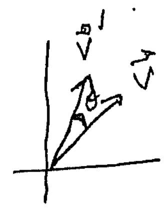
$\hat{e}'_n = \underbrace{\sum_m \hat{e}_m \hat{e}_m}_{\mathbb{1}} \cdot R\hat{e}_n = \sum_m \hat{e}_m R_{mn}$ N.B. Thus $\hat{e}'_n = R^T_{nm}\hat{e}_m$

$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$R = \sum_{m,n} \hat{e}_m R_{mn} \hat{e}_n$

~~But~~ we call this R not R^T in adv. mech. notes

Active rotation of $\vec{V}$



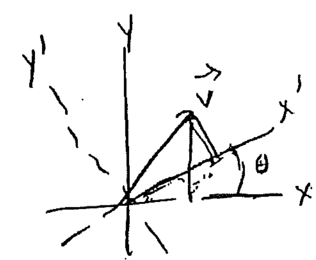
Define $V_n = \hat{e}_n \cdot \vec{V} \Rightarrow \vec{V} = \sum_n \hat{e}_n V_n$

$\vec{V}' = R\vec{V} = \sum_n (R\hat{e}_n) V_n = \sum_n \hat{e}'_n V_n$ compare

Define $\boxed{V'_n = \hat{e}_n \cdot \vec{V}'} \Rightarrow \vec{V}' = \sum_n \hat{e}_n V'_n$

$V'_n = \hat{e}_n \cdot R\vec{V} = \sum_m \hat{e}_n \cdot R\hat{e}_m \, \hat{e}_m \cdot \vec{V} = \sum_m R_{nm} V_m$

Passive rotation



Let $\vec{V}$ be fixed and define

$V_n = \hat{e}_n \cdot \vec{V},$

$\boxed{V'_n = \hat{e}'_n \cdot \vec{V}}$ (differs from above)

$V'_n = \sum_m \hat{e}'_n \cdot \hat{e}_m \, \hat{e}_m \cdot \vec{V} = \sum_m R_{mn} V_m = \sum (R^T)_{nm} V_m \qquad R^T = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

(1-14-24)

Backgrnd material

I worked this out for my own satisfaction!

P3120
Spr 2024

Let R be a real, orthogonal matrix: $R^T R = \mathbb{1}$

Consider its eigenvalues $Rv = \lambda v$ for nonzero v

$$v^\dagger R^\dagger = \lambda^* v^\dagger$$

$$v^\dagger \underbrace{\underbrace{R^\dagger R}_{R^T R}}_{1} v = |\lambda|^2 v^\dagger v$$

Since $v^\dagger v$ is manifestly nonzero (if v is nonzero), $\lambda = e^{i\theta}$

(Observe that $v^T R^T = \lambda v^T \Rightarrow v^T v = \lambda^2 v^T v$

so either $\lambda = \pm 1$ or $v^T v = 0$.)

Restrict now to 3×3 matrices.

Let $\hat{n}$ be the eigenvector of R w/ eigenvalue 1

$$R\hat{n} = \hat{n}$$

Then $R = e^{-i\phi \hat{n}\cdot\vec{L}}$ ← NB. This corresponds to an active rotation through ϕ (passive would have opp. sign)

where $(L^a)_{mn} = i\epsilon^{man}$

This is because $\underbrace{(\hat{n}\cdot\vec{L})\hat{n}}_{i n^a \epsilon^{man} n^n} = 0 \Rightarrow R\hat{n} = \hat{n}$

The other two eigenvectors are complex, & eigenvalues are $e^{\pm i\phi}$

e.g. $\hat{n} = \hat{z}$

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ +\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} = e^{\mp i\phi} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

Background

$R^T R = 1$

$R = e^r \quad \Rightarrow \quad r^T = r$

R can be expressed as a rotation about $\vec{v}$ through angle θ

$R v = v$ solve for v to find axis of rotation (normalize it)

$\det(R - \lambda) = 0 \quad \Rightarrow \quad \lambda = 1, e^{i\theta}, e^{-i\theta}$ for angle of rotation

Then $r_{mn} = \theta \epsilon_{mnl} v_l$ ($\exists$ sign ambiguity in both θ + v)

Eg. $R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$Rv = v \quad \Rightarrow \quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$[(\lambda - \cos\theta)^2 + \sin^2\theta][\lambda - 1] = 0 \qquad \lambda = e^{i\theta}, e^{-i\theta}, 1$

$r = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(1-12-24)

Background

Fetter-Walecka: Theoretical mechanics, §7 (p. 33)

Tong: Classical Dynamics, §3

Let $\hat{e}^k$ be orthonormal body-fixed axes: $\hat{e}^k \cdot \hat{e}^l = \delta_{kl}$

Goal: show $\frac{d\hat{e}^k}{dt} = \vec{\omega} \times \hat{e}^k$

Proof:

Let $\frac{d\hat{e}^k}{dt} = \omega_{kl}\,\hat{e}^l \quad \Rightarrow \quad \omega_{kl} = \frac{d\hat{e}^k}{dt} \cdot \hat{e}^l$

orthonormality $\Rightarrow \omega_{kl} = -\omega_{lk}$

Define $\omega_{kl} = \epsilon_{klm}\,\omega_m \quad \Rightarrow \frac{d\hat{e}^k}{dt} = \epsilon_{klm}\,\omega_m\,\hat{e}^l$

$\omega_m = \frac{1}{2}\,\epsilon_{mkl}\,\omega_{kl} = \frac{1}{2}\,\epsilon_{mkl}\,\frac{d\hat{e}^k}{dt} \cdot \hat{e}^l$

Define $\vec{\omega} = \omega_m \hat{e}^m$ ie ω_m = body-fixed components of $\vec{\omega}$

Then $\vec{\omega} \times \hat{e}^k = \omega_m\,\hat{e}^m \times \hat{e}^k = \omega_m\,\epsilon_{mkl}\,\hat{e}^l = \frac{d\hat{e}^k}{dt}$ (QED)

Note $R_{ki} = \hat{e}^k_i \qquad R_{ki} R_{li} = \delta_{kl}$

(cf Tong §3.1.1) $\omega_{kl} = \left(\frac{d}{dt} e^k_i\right) e^l_i = \frac{dR_{ki}}{dt} R_{li} = -R_{ki} \frac{dR_{li}}{dt} = -\omega_{lk}$

$\left(\frac{d\vec{A}}{dt}\right)_{space} = \sum \frac{dA_i}{dt}\hat{x}_i = \sum\left(\frac{dA'_k}{dt}\hat{e}_k + A'_k \frac{d\hat{e}_k}{dt}\right)$

$= \sum\left(\frac{dA'_k}{dt}\right)\hat{e}_k + \vec{\omega} \times \sum A'_k \hat{e}_k$

$\boxed{\left(\frac{d\vec{A}}{dt}\right) = \left(\frac{d\vec{A}}{dt}\right)_{\dots} + \vec{\omega} \times \vec{A}}$

Background

$$\hat{e}^k = R_{km}\,\hat{x}^m \quad \Rightarrow \quad \hat{x}^m = (R^{-1})_{ml}\,\hat{e}^l$$

$$\frac{d\hat{e}^k}{dt} = \dot{R}_{km}\,\hat{x}^m = \dot{R}_{km}(R^{-1})_{ml}\,\hat{e}^l$$

$$= (\dot{R}R^{-1})_{kl}\,\hat{e}^l$$

Define $\omega_{kl} \equiv (\dot{R}R^{-1})_{kl} = (\dot{R}R^T)_{kl} \Rightarrow \omega^T = R\dot{R}^T$

$$RR^T = 1 \Rightarrow \dot{R}R^T + R\dot{R}^T = 0$$

$$\omega + \omega^T = 0 \quad \Rightarrow \omega_{kl} = \epsilon_{kla}\,\omega_a$$

$$\frac{d\hat{e}^k}{dt} = \omega_{kl}\,\hat{e}^l$$

$$\hat{e}^k(t+dt) = (\delta_{kl} + \omega_{kl}dt)\,\hat{e}^l(t)$$

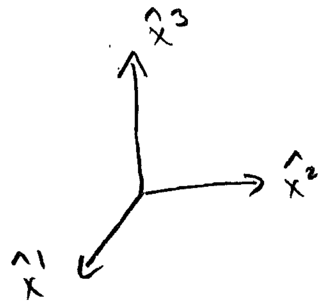
$$(L^a)^{mn} = i\epsilon^{man}$$

$$= (\delta_{kl} + \underbrace{\epsilon_{kla}\omega_a dt}_{+i\omega_a dt (L^a)^{kl}})\,\hat{e}^l(t)$$

$$= \underbrace{e^{i\omega_a dt L^a}}_{\text{passive}}\,\hat{e}^l(t)$$

Recall $R = e^{-i\phi\hat{n}\cdot\vec{\tau}}$ = active rotation

$e^{i\phi\hat{n}\cdot\vec{\tau}}$ = passive rotation



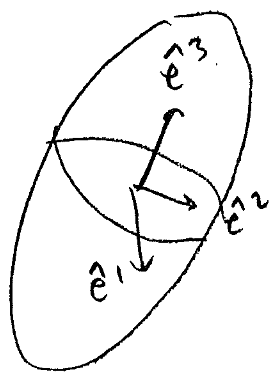
Let $\hat{x}^m$ be Cartesian unit vectors of an inertial reference frame

"space fixed frame"

orthonormal: $\hat{x}^m \cdot \hat{x}^n = \delta_{mn}$

Arbitrary vector $\vec{A} = \sum_m A_m \hat{x}^m$

$$A_m = \hat{x}^m \cdot \vec{A} = \hat{x}^m \cdot \sum A_n \hat{x}^n = \sum_n A_n \delta_{mn} = A_m \checkmark$$



Let $\hat{e}^k$ be principal axes of a rigid body

orthonormal $\hat{e}^k \cdot \hat{e}^l = \delta_{kl}$

These can be taken as basis vectors of a coordinate system attached to object

"body fixed frame"

<u>Not</u> an inertial reference frame because object rotates + so does the body fixed frame.

Advantage: inertia tensor is diagonal in body fixed frame + also doesn't change in time (object not changing wrt. body fixed frame)

Arbitrary vector $\vec{A} = \sum A'_k \hat{e}^k$

$A'_k = \hat{e}^k \cdot \vec{A}$

use primes now for components in body fixed frame [Marion + Thornton, p37]

[Consider using another notation than primes, maybe tilde?]

Let e^k_m $(m=1,2,3)$ be components of $\hat{e}^k$ in space fixed frame

$$\hat{e}^k = \sum_m e^k_m \hat{x}^m \qquad e^k_m = \hat{e}^k \cdot \hat{x}^m$$

We can use e^k_m to relate A_m and A'_k

$$A'_k = \hat{e}^k \cdot \vec{A} = \sum_m e^k_m \hat{x}^m \cdot \vec{A} = \sum_m e^k_m A^m$$

Define matrix R of matrix of elements $R_{km} \equiv e^k_m$ (row k, column m)

$\therefore R$ here = transpose of R used for active transf.

[NB - passive transformation]

$$A'_k = \sum R_{km} A_m \qquad \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = R \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Also

$$A_m = \hat{x}^m \cdot \vec{A} = \hat{x}^m \cdot \sum_k A'_k \hat{e}^k = \sum_k e^k_m A'_k$$

$$= \sum R_{km} A'_k = \sum R^T_{mk} A'_k \qquad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = R^T \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = R^T R \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad \text{for any } \vec{A} \quad \text{so} \quad R^T R = \mathbb{1}$$

ie R is an <u>orthogonal matrix</u>.

⇒ the transformation from $\hat{x}^m$ to $\hat{e}^k$ is a rotation. (will later characterize by Euler angles)

If both $\hat{x}^m$ & $\hat{e}^k$ are right handed, then $\det R = 1$ (because $\det R = \pm 1$ but right handed crd sys are continuously connected)

Group of orthogonal matrices of determinant 1 is called $SO(3)$

[N.B. do not introduce summation convention because apparently violated by $I_k e^k_m$!]

Recall: $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

In space-fixed frame: $L_m = \sum_n I_{mn} \omega_n$

In body-fixed frame: $L'_k = \sum_\ell I'_{k\ell} \omega'_\ell$ (*)

We know $L'_k = \sum R_{\underline{\underline{km}}} L_m$ and $\omega'_k = \sum R_{km} \omega_m$. Also $\omega_n = \sum R_{\ell n} \omega'_\ell$

because $\vec{L}$ + $\vec{\omega}$ are vectors (1^{st} rank tensors)

How are $I'_{k\ell}$ and I_{mn} related?

$$L'_k = \sum_m R_{km} L_m = \sum_m \sum_n R_{km} I_{mn} \omega_n = \sum_m \sum_n \sum_\ell R_{km} I_{mn} R_{\ell n} \omega'_\ell$$

Comparing w/ (*) we see

$$I'_{k\ell} = \sum_m \sum_n \underline{\underline{R_{km}}}\, \underline{\underline{R_{\ell n}}}\, I_{mn}$$

$\overleftrightarrow{I}$ transforms as a 2^{nd} rank tensor

E.g. a $\underline{3^{rd}}$ rank tensor Ω_{mnp} transforms as

$$\Omega'_{k\ell i} = \sum_m \sum_n \sum_p R_{km} R_{\ell n} R_{ip} \Omega_{mnp}$$

Compute inertial tensor in body fixed frame

$$I'_{k\ell} = \sum_m \sum_n R_{km} R_{\ell n} I_{mn} \qquad (*)$$

$$= \sum_m \sum_n e^k{}_m I_{mn} e^\ell{}_n$$

Recall $\sum_n I_{mn} e^\ell{}_n = I_\ell e^\ell{}_m$ defines principal axes & moments

$$I'_{k\ell} = \sum_m e^k{}_m I_\ell e^\ell{}_m$$

$$= I_\ell \, \hat{e}^k \cdot \hat{e}^\ell$$

$$= I_\ell \, \delta_{k\ell}$$

$$\overleftrightarrow{I}' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

diagonal in body-fixed frame

[makes sense because object balanced wrt. principal axes]

Can write

$$\overleftrightarrow{I}' = R \overleftrightarrow{I} R^{-1}$$

← similarity transformation
[check that agrees w/ (*)]

Thm: any symmetric real matrix can be diagonalized by an orthogonal similarity tx

[com: any hermitian ... unitary]

Summarize

$$\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega} \xrightarrow[\text{space fixed}]{} L_m = \sum_n I_{mn}\omega_n$$

$$\vec{L}' = R\vec{L} = R\overleftrightarrow{I}\cdot\vec{\omega}$$
$$= R\overleftrightarrow{I}R^{-1}R\vec{\omega} \xrightarrow[\text{body fixed}]{} L'_k = I_k\omega'_k$$
$$= \overleftrightarrow{I}'\cdot\vec{\omega}'$$

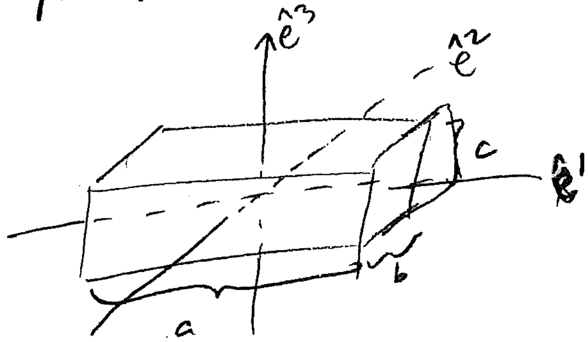
$$T = \tfrac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} \longrightarrow T = \frac{1}{2}\sum_m\sum_n \omega_m I_{mn}\omega_n$$

$$= \frac{1}{2}\vec{\omega}R^{-1}R\overleftrightarrow{I}R^{-1}R\vec{\omega}$$
$$= \frac{1}{2}\vec{\omega}'\cdot\overleftrightarrow{I}'\cdot\vec{\omega}' \longrightarrow T = \frac{1}{2}\sum_k I_k\omega_k'^2$$

$$\left(\begin{array}{l}\vec{\omega}' = R\vec{\omega}\\ \vec{\omega}'^T = \vec{\omega}^T R^T = \vec{\omega}^T R^{-1}\end{array}\right)$$

~~58~~
~~57~~
56

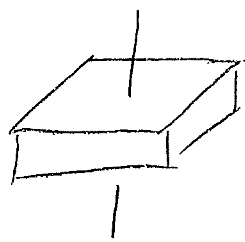
If object has some obvious symmetries principal axes should be obvious



(brick)

$$I = \frac{1}{12} M \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2+a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

If $a = b$:



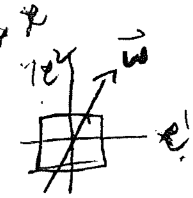
(square plate)

$$I = \frac{1}{12} M \begin{bmatrix} a^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & 2a^2 \end{bmatrix}$$

122 phymat pointed out I should use $\hat{e}^i$ not $\hat{x}^i$

$I_1 = I_2 \Rightarrow$ not only $\hat{e}^1$ + $\hat{e}^2$ but any linear comb. thereof is a principle axis

$$\overleftrightarrow{I} \cdot (\alpha \hat{e}^1 + \beta \hat{e}^2) = \alpha \overleftrightarrow{I} \cdot \hat{e}^1 + \beta \overleftrightarrow{I} \cdot \hat{e}^2 = \alpha I_1 \hat{e}^1 + \beta I_2 \hat{e}^2$$
$$= I_1 (\alpha \hat{e}^1 + \beta \hat{e}^2)$$



(Any axis in xy plane is a principal axis)
(this is why proof of orthogonality fails)

Degenerate eigenvalue $\Rightarrow$ any linear comb. of corresponding eigenvectors is an eigenvector of same eigenvalue

For a cube $\qquad \overleftrightarrow{I} = \frac{1}{6} m a^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↓

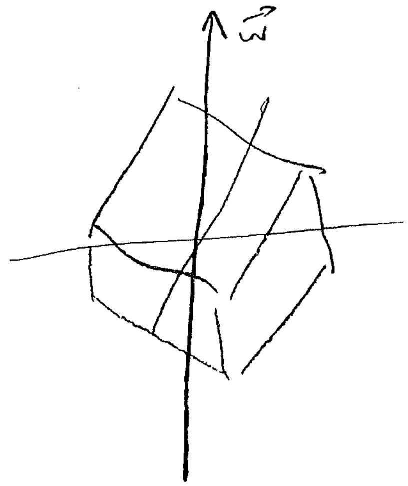
$I_1 = I_2 = I_3$

↓

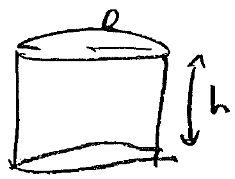
$\Rightarrow$ any vector is an eigenvector

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{6} m a^2 \vec{\omega}$$ for any $\vec{\omega}$ through center of cube

$\Rightarrow$ rotation about any axis is dynamically stable [see ahead]



cylinder:



For what value of h/R is cylinder dynamically stable about any axis?

[ask next class]

$$\frac{1}{2} m R^2 = \frac{1}{12} m h^2 + \frac{1}{4} m R^2$$

$$\Rightarrow h = \sqrt{3} R$$