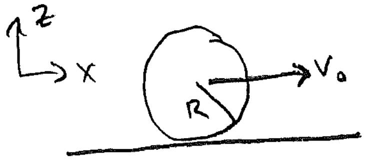
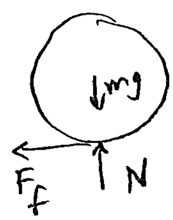
Sliding bowling ball



Hits ground w/ speed v_0 and no rotation.
When does it begin to roll?
What is its speed?
How far has it slid?



$F_f = \mu N = \mu mg$ $\qquad$ μ = coeff of <u>kinetic</u> friction

Net force $\vec{F} = -\mu mg\,\hat{x}$ (until it begins to roll!)

$\vec{F}^{ext} = m\vec{a}_{cm}$ $\qquad$ $a_x = \dfrac{F_x}{m} = -\mu g$

$v_x = v_0 - \mu g t$

$x = x_0 + v_0 t - \frac{1}{2}\mu g t^2$

$\vec{L}' = I\vec{\omega}$

$\dfrac{d\vec{L}'}{dt} = I\vec{\alpha} = \vec{\tau}^{ext}$

Torque about cm $\quad \vec{\tau} = R F_f\,\hat{y} = R\mu mg\,\hat{y}$

$\vec{\tau} = I\vec{\alpha}$ $\qquad$ $\alpha_y = \dfrac{\tau_y}{I} = \dfrac{R\mu mg}{I} = \dfrac{5}{2}\dfrac{\mu g}{R}$

$\omega_y = \cancel{\omega_{0y}}^{\,0} + \dfrac{5}{2}\dfrac{\mu g}{R}t$

Ball starts to roll when $v_x = R\omega_y$ (point of contact not moving)

$v_0 - \mu g t = \frac{5}{2}\mu g t$

$t_{roll} = \dfrac{2}{7}\dfrac{v_0}{\mu g}$

Speed of ball $v_x = R\left(\dfrac{5}{2}\dfrac{\mu g}{R}\right)\left(\dfrac{2}{7}\dfrac{v_0}{\mu g}\right) = \dfrac{5}{7}v_0$

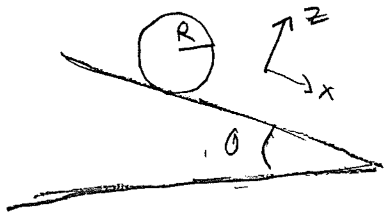
It has slid $v_0 t - \frac{1}{2}\mu g t^2 = \dfrac{2}{7}\dfrac{v_0^2}{\mu g} - \dfrac{2}{49}\dfrac{v_0^2}{\mu g} = \dfrac{12}{49}\dfrac{v_0^2}{\mu g}$

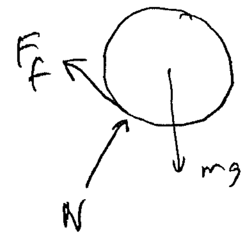
What happens after t roll?

[$F_f = 0$, $\vec{v}$ + $\vec{\omega}$ are constant]

?

Cylinder rolling down incline





Net force

$$F_x = mg \sin\theta - F_f = m a_x$$

[what is F_f? $\mu mg\cos\theta$? No!] F_f is what it needs to be to keep ball rolling provided $F_f \le \mu mg\cos\theta$.

Torque about cm [accelerating]

$$\tau_y = R F_f = I \alpha_y$$

Assume rolling w/o sliding:

$$v_x = R\omega_y$$
$$a_x = R\alpha_y$$

$$\Rightarrow mg\sin\theta - F_f = mR\alpha_y = \frac{mR^2 F_f}{I}$$

$$F_f = \frac{mg\sin\theta}{\left(1 + \frac{mR^2}{I}\right)}$$

$$a_x = g\sin\theta - \frac{F_f}{m} = \frac{g\sin\theta}{\left(\frac{I}{mR^2} + 1\right)}$$

e.g. $I = \frac{1}{2}mR^2$
$\Rightarrow a = \frac{2}{3} g\sin\theta$

Check whether cylinder will roll or slide

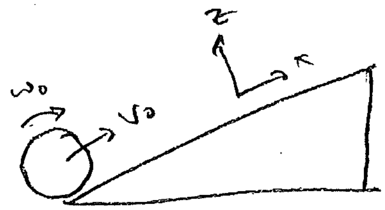
$$F_f \le \mu mg\cos\theta$$

$$\tan\theta \le \mu\left(1 + \frac{mR^2}{I}\right)$$

eg disk $I = \frac{1}{2}mR^2$

$\tan\theta \le 3\mu$

otherwise will start sliding and also begin to roll.



Initially rolling $\Rightarrow$ $\omega_0 = \frac{v_0}{R}$

$$F_x = -mg\sin\theta + F_f = ma_x$$

$$v = v_0 + \left(\frac{F_f}{m} - g\sin\theta\right)t$$

$$x = v_0 t + \frac{1}{2}\left(\frac{F_f}{m} - g\sin\theta\right)t^2 \qquad \Rightarrow \quad h = x\sin\theta$$

$$\tau_y = -RF_f$$

$$\omega = \omega_0 - \frac{RF_f}{I}t$$

Will continue to roll w/o slipping provided $\frac{F_f}{m} - g\sin\theta = -\frac{R^2F_f}{I}$

$$\Rightarrow \left(1 + \frac{mR^2}{I}\right)F_f = mg\sin\theta$$

$$F_f = \frac{mg\sin\theta}{1 + \frac{mR^2}{I}}$$

(a) $F_f \leq \mu mg\cos\theta \quad \Rightarrow \quad \boxed{\mu_{crit} = \frac{\tan\theta}{1 + \frac{mR^2}{I}}}$

(b) $\mu > \mu_{crit} \Rightarrow a = \frac{F_f}{m} - g\sin\theta = \frac{g\sin\theta}{1 + \frac{mR^2}{I}} - g\sin\theta = \frac{-mR^2}{I + mR^2}g\sin\theta$

$v = v_0 + at = 0$ when $t = -\frac{v_0}{a} = \frac{v_0(I + mR^2)}{mR^2 g\sin\theta}$

$$\Rightarrow x = v_0\left(-\frac{v_0}{a}\right) + \frac{1}{2}a\left(-\frac{v_0}{a}\right)^2 = -\frac{v_0^2}{2a} = \frac{v_0^2(I + mR^2)}{2g\sin\theta\, mR^2}$$

$$= \frac{v_0^2}{g\sin\theta}\left(\frac{I}{mR^2} + 1\right)$$

$$h = x\sin\theta = \boxed{\frac{v_0^2}{2g}\left(\frac{I}{mR^2} + 1\right)}$$

(c) If $\mu < \mu_{crit} \Rightarrow F_f = \mu m g \cos\theta$

$$v = v_0 + (\mu g \cos\theta - g \sin\theta) t$$

$$\Rightarrow t = \frac{v_0}{g(\sin\theta - \mu\cos\theta)} \qquad \Rightarrow x = -\frac{v_0^2}{2a} = \frac{v_0^2}{2g(\sin\theta - \mu\cos\theta)}$$

$$\boxed{h = \frac{v_0^2}{2g(1 - \mu\cot\theta)}}$$

if $\mu = \mu_{crit}$

$$\frac{v_0^2}{2g\left(1 - \frac{1}{1+\frac{mR^2}{I}}\right)} = \frac{v_0^2}{2g}\frac{(I + mR^2)}{(mR^2)}$$

(d)

$$\omega = \omega_0 - \frac{R\mu m g \cos\theta}{I} t$$

$$= \frac{v_0}{R} - \frac{v_0 \mu m g \cos\theta R}{Ig(\sin\theta - \mu\cos\theta)}$$

$$= \frac{v_0}{R}\left[1 - \frac{\mu m R^2 \cos\theta}{I(\sin\theta - \mu\cos\theta)}\right]$$

$$= \frac{v_0}{R}\frac{I(\sin\theta - \mu\cos\theta) - \mu m R^2 \cos\theta}{I(\sin\theta - \mu\cos\theta)}$$

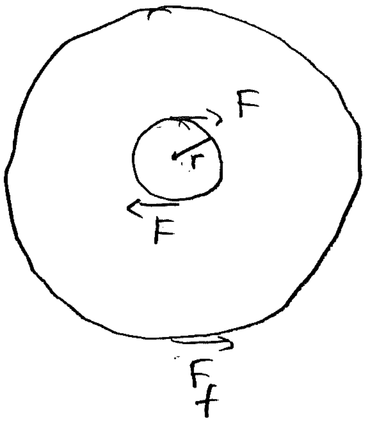
$$= \frac{v_0}{R}\frac{\left(\sin\theta - \mu\left(1 + \frac{mR^2}{I}\right)\cos\theta\right)}{(\sin\theta - \mu\cos\theta)} = \frac{v_0}{R}\left[\frac{\tan\theta - \mu\left[1 + \frac{mR^2}{I}\right]}{\tan\theta - \mu}\right]$$

If $\mu = 0 \Rightarrow \omega = \frac{v_0}{R}$ ✓

If $\mu = \mu_{crit} \Rightarrow \omega = 0$ ✓

Work done to accelerate a wheel

(no slipping)



- $F_f = m a_{cm}$ net force
- $2rF - RF_f = I\alpha$ net torque
- $a = R\alpha$ rolling w/o slipping

$$\Downarrow$$

$$\frac{F_f}{m} = R\,\frac{(2rF - RF_f)}{I} \quad \cancel{\Rightarrow F_f}$$

$$2mRrF = (I + mR^2)F_f \Rightarrow \cancel{F_f = \frac{2mRrF}{(I+mR^2)}}$$

Frictional force does no work because point of contact moves perpendicular to force. All work is done by the torque applied to the axle

$$\Delta W = \tau \Delta\phi$$

$$\Delta W = \underline{2Fr\Delta\phi} = 2rF\omega\Delta t = \left(\frac{I + mR^2}{mR}\right)F_f\,\omega\,\Delta t$$

Change in kinetic energy

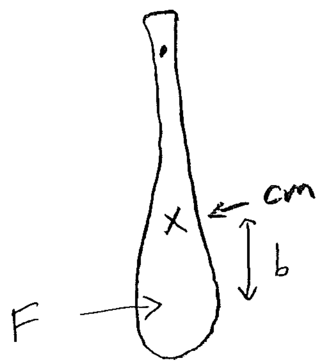
$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \underset{\substack{\uparrow \\ v=\omega R}}{=} \frac{1}{2}(I + mR^2)\,\omega^2$$

$$\Delta K = (I + mR^2)\,\omega\,\Delta\omega$$

$$= (I + mR^2)\,\omega\,\alpha\,\Delta t \qquad \text{but } \alpha = \frac{a}{R} = \frac{F_f}{mR}$$

$$= \left(\frac{I + mR^2}{mR}\right)\omega\,F_f\,\Delta t,$$ agrees w/ above

Center of percussion



A force F is applied for a short time interval Δt at a distance b below the cm. of some object.

b = impact parameter.

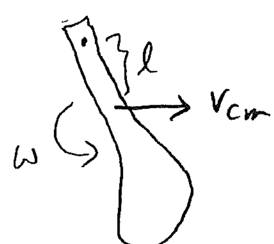
$$F = m \frac{\Delta V_{cm}}{\Delta t}$$

If object initially at rest, after the impulse $V_{cm} = \frac{F \Delta t}{m}$

The force results in a torque about cm $\tau = bF$

$$\tau = I' \frac{\Delta \omega}{\Delta t}$$

After the impulse, object rotates w/ $\omega = \frac{\tau \Delta t}{I'} = \frac{bF\Delta t}{I'}$



Speed of a point a distance l above cm is $V_{cm} - l\omega$

This point remains stationary if

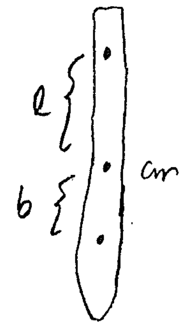
$$l\omega = V_{cm}$$

ie acts as a pivot

$$\frac{l\, bF\Delta t}{I'} = \frac{F\Delta t}{m}$$

$$b = \frac{I'}{ml}$$

Point of impact called center of percussion



Distance of impact point from pivot

$$b + l = \frac{(I' + ml^2)}{ml} = \frac{I}{ml}$$

where I = moment of inertia about pivot

This is same as radius of gyration, k.

For a rod of length $L = 2l$, $I = \frac{1}{3}ML^2$

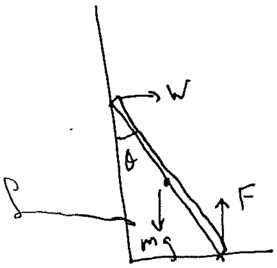
$$\therefore k = \frac{2}{3}L$$

(For a bat, a bit larger because heavier at the end)

beyond the cm

Ladder + wall

Consider a ladder on a frictionless floor leaning against a frictionless wall, starting at rest at $\theta = \theta_0$



Because the wall + floor forces do no work since the point of contact moves transversely to the force we can use energy conservation

[assume ladder in contact w/ wall]

$$E = T_{cm} + T_{rot} + U_{grav}$$

$$= \underbrace{\frac{m\ell^2}{8}\dot\theta^2 + \frac{m\ell^2}{24}\dot\theta^2}_{\frac{m\ell^2}{6}\dot\theta^2} + \frac{mg\ell}{2}\cos\theta = \frac{mg\ell}{2}\cos\theta_0$$

see below for $\vec v_{cm}$

$$\Rightarrow \dot\theta^2 = 3\frac{g}{\ell}(\cos\theta_0 - \cos\theta)$$

— Alternatively we can use forces and torques: [assume ladder remains in contact w/ wall]

$$x_{cm} = \frac{\ell}{2}\sin\theta$$

$$\dot x_{cm} = \frac{\ell}{2}\cos\theta\,\dot\theta$$

$$\ddot x_{cm} = -\frac{\ell}{2}\sin\theta\,\dot\theta^2 + \frac{\ell}{2}\cos\theta\,\ddot\theta$$

$$z_{cm} = \frac{\ell}{2}\cos\theta$$

$$\dot z_{cm} = -\frac{\ell}{2}\sin\theta\,\dot\theta$$

$$\ddot z_{cm} = -\frac{\ell}{2}\cos\theta\,\dot\theta^2 - \frac{\ell}{2}\sin\theta\,\ddot\theta$$

$$F_x = W = \frac{m\ell}{2}\left[\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\right]$$

$$F_z = F - mg = \frac{m\ell}{2}\left[-\sin\theta\,\ddot\theta - \cos\theta\,\dot\theta^2\right]$$

$$\tau = \frac{\ell}{2}F\sin\theta - \frac{\ell}{2}W\cos\theta = \frac{1}{12}m\ell^2\ddot\theta$$

($\dot\theta^2$ cancels out!)

Substitute 1st 2 eqns into last to get

$$-\frac{m\ell^2}{4}\ddot\theta + mg\frac{\ell}{2}\sin\theta = \frac{1}{12}m\ell^2\ddot\theta$$

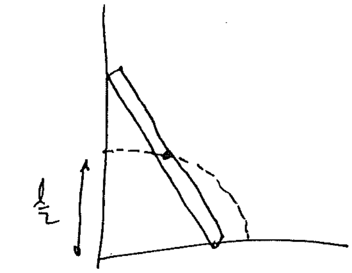
$$\ddot\theta = \frac{3g}{2\ell}\sin\theta$$

$$\dot\theta^2 = -\frac{3g}{\ell}\cos\theta + \text{const} \quad (\text{as above})$$

But this problem poses an interesting paradox

The cm follows a quarter circle

since $x_{cm}^2 + z_{cm}^2 = \frac{l^2}{4}$



Note that when the ladder hits the ground, the horizontal velocity (& ∴ momentum) is zero.
But the initial horizontal momentum is also zero.
How is this possible if the wall imparts a positive impulse?

Let's solve for W and F

$$W = \frac{ml}{2}\left[\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\right]$$

$$= \frac{ml}{2}\left[\cos\theta\left(\frac{3g}{2l}\sin\theta\right) - \sin\theta\left(-\frac{3g}{l}\cos\theta + \frac{3g}{l}\cos\theta_0\right)\right]$$

$$= \frac{mg}{4}\left[9\cos\theta - 6\cos\theta_0\right]\sin\theta$$

$$F = mg - \frac{ml}{2}\left[\sin\theta\left(\frac{3g}{2l}\sin\theta\right) + \cos\theta\left(\frac{3g}{l}\right)(\cos\theta_0 - \cos\theta)\right]$$

$$= mg - \frac{mg}{4}\left[3\underbrace{\sin^2\theta}_{1-\cos^2\theta} - 6\cos^2\theta + 6\cos\theta_0\cos\theta\right]$$

$$= \frac{mg}{4}\left[1 + (9\cos\theta - 6\cos\theta_0)\cos\theta\right]$$

Note that W goes negative if $\cos\theta < \frac{2}{3}\cos\theta_0$
so before the ladder hits the ground, it loses contact w/ wall.
To remain in contact, W would have to act leftward, &
the total impulse from θ_0 to $\theta = \frac{\pi}{2}$ would be zero
(simply integrate $\int W\,dt = \int dp_x = p_x\Big|_{\theta_0}^{\pi/2} = 0$, as calc'd above)

What is the horizontal speed of ladder when it hits the ground?

[After I solved the paradox + problem, I searched + found this problem in Morin 8.3]
↓
$\dot{\theta}_0 = 0$.

~~After ladder~~

Ladder loses contact w/wall when

$$W = \frac{ml}{2}\left[\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right] = 0$$

Since $\ddot{\theta} = \frac{3g}{2l}\sin\theta$

and $\dot{\theta}^2 = 3\frac{g}{l}(\cos\theta_0 - \cos\theta)$

then implies

$$\frac{1}{2}\sin\theta\cos\theta = \sin\theta(\cos\theta_0 - \cos\theta)$$

$$\cos\theta = \frac{2}{3}\cos\theta_0$$

ie

$$\Rightarrow \dot{\theta}^2 = 3\frac{g}{l}\left(\frac{1}{3}\cos\theta_0\right) = \frac{g}{l}\cos\theta_0$$

At this instant $\dot{x}_{cm} = \frac{l}{2}\cos\theta\,\dot{\theta}$

$$= \frac{l}{2}\left(\frac{2}{3}\cos\theta_0\right)\sqrt{\frac{g}{l}\cos\theta_0}$$

$$= \frac{1}{3}\cos\theta_0\sqrt{gl\cos\theta_0} \quad \Rightarrow \frac{1}{2}m\dot{x}_{cm}^2 = \frac{mgl\cos^3\theta_0}{18}$$

Thereafter there is no force in x direction so velocity is constant!

$$\dot{x}_{cm} = \sqrt{\frac{gl\cos^3\theta_0}{9}}$$

[check: if $\cos\theta_0 = 0.9$, $\dot{x}_{cm} = 0.285\sqrt{gl}$]

$\Rightarrow E_{initial} = \frac{mgl}{2}\cos\theta_0$, $\frac{K}{E} = \frac{1}{9}\cos^2\theta_0$

The energy of the ladder is

$$E = T_{cm} + T_{rot} + U_{grav}$$

$$= \frac{1}{2} m \dot{x}_{cm}^2 + \frac{1}{2} m \dot{z}_{cm}^2 + \frac{1}{24} m l^2 \dot{\theta}^2 + \frac{mgl}{2}\cos\theta = \frac{mgl}{2}\cos\theta_0$$

While the ladder remains in contact w/ the wall

$$x_{cm} = \frac{l}{2}\sin\theta \qquad z_{cm} = \frac{l}{2}\cos\theta$$

$$\dot{x}_{cm} = \frac{l}{2}\dot{\theta}\cos\theta \qquad \dot{z}_{cm} = -\frac{l}{2}\dot{\theta}\sin\theta$$

and so

$$\frac{1}{2} m v_{cm}^2 = \frac{1}{8} \cdot m l^2 \dot{\theta}^2 \quad \text{so}$$

$$E = \frac{1}{6} m l^2 \dot{\theta}^2 + \frac{mgl}{2}\cos\theta = \frac{mgl}{2}\cos\theta_0$$

$$\Rightarrow \dot{\theta}^2 = \frac{3g}{l}(\cos\theta_0 - \cos\theta)$$

At the instant the ladder leaves the wall ($\cos\theta = \frac{2}{3}\cos\theta_0$, $\dot{\theta}^2 = \frac{g}{l}\cos\theta_0$)

$$\dot{x}_{cm} = \frac{1}{3}\cos\theta_0 \sqrt{gl\cos\theta_0}$$

and remains so thereafter so

$$E = \frac{1}{18} mgl\cos^3\theta_0 + \frac{1}{8} m l^2 \dot{\theta}^2 \sin^2\theta + \frac{1}{24} m l^2 \dot{\theta}^2 + \frac{mgl}{2}\cos\theta = \frac{mgl}{2}\cos\theta_0$$

so

$$\frac{1}{24} m l^2 (1 + 3\sin^2\theta)\dot{\theta}^2 = \frac{mgl}{2}\left(\cos\theta_0 - \cos\theta - \frac{1}{9}\cos^3\theta_0\right)$$

$$\left[\dot{\theta}^2 = \frac{12}{(1+3\sin^2\theta)} \frac{g}{l}\left(\cos\theta_0\left[1 - \frac{1}{9}\cos^2\theta_0\right] - \cos\theta\right)\right]$$

differentiate

$$\frac{1}{12} m l^2 (1 + 3\sin^2\theta)\ddot{\theta} + \frac{1}{4} m l^2 \sin\theta\cos\theta\,\dot{\theta}^2 = \frac{mgl}{2}\sin\theta$$

which precisely agrees w/ →

once bar leaves wall :

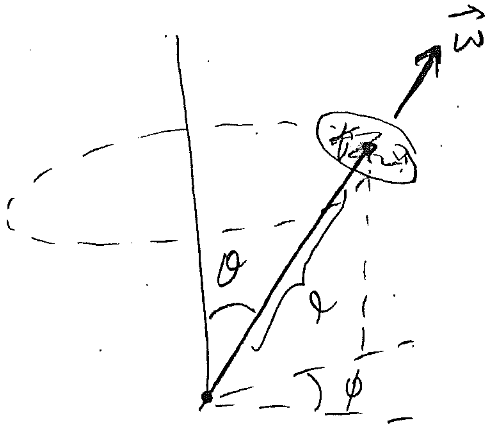
$$\tau = \frac{l}{2}\sin\theta\left[mg - \frac{ml}{2}\sin\theta\,\ddot{\theta} - \frac{ml}{2}\cos\theta\,\dot{\theta}^2\right] = \frac{1}{12}ml^2\ddot{\theta}$$

$$= \left(\frac{1}{12}ml^2 + \frac{1}{4}ml^2\sin^2\theta\right)\ddot{\theta} + \frac{ml^2}{4}\sin\theta\cos\theta\,\dot{\theta}^2 = \frac{mgl}{2}\sin\theta$$

$$\frac{mgl}{2}\sin\theta = \left(\frac{1}{12}ml^2 + \frac{1}{4}ml^2\sin^2\theta\right)\ddot{\theta} + \frac{ml^2}{4}\sin\theta\cos\theta\,\dot{\theta}^2$$

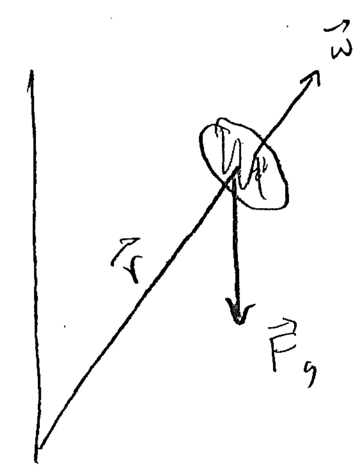
Simplified treatment of precessing top

Consider a top with its tip fixed spinning w/ angular speed $\vec{\omega}$



The external gravitational torque will cause it to precess around a vertical line w/ precession frequency $\dot{\phi}$ given approximately by $\frac{mgl}{I\omega}$

We'll derive this now in a simplified treatment and later more rigorously



$$\vec{L} = I\vec{\omega}$$

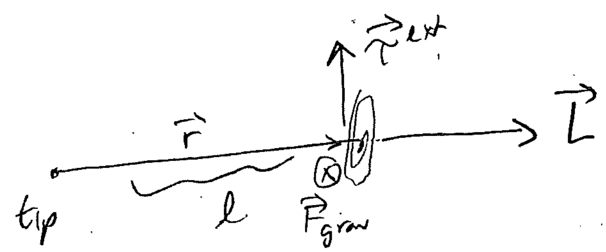
$$\vec{\tau}^{\,ext} = \vec{r} \times \vec{F}_g$$

($\vec{L}$ + $\vec{\tau}$ both defined wrt. fixed tip, force on tip exerts no torque.)

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{\,ext}$$

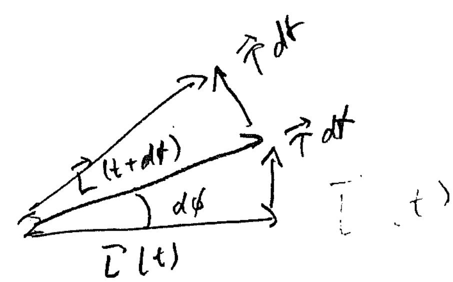
First consider $\theta = \frac{\pi}{2}$

View from overhead



$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext} \Rightarrow \vec{L}(t+dt) = \vec{L}(t) + \vec{\tau}\, dt$$

torque causes $\vec{L}$ to change direction ($\therefore$ top itself)



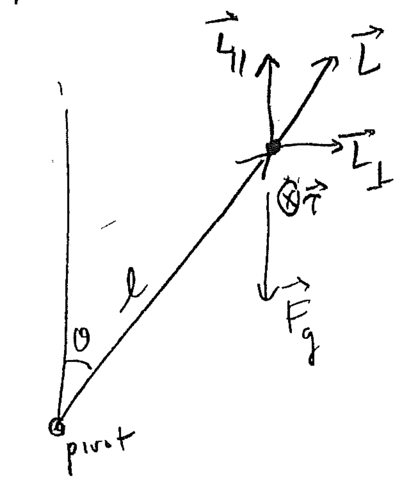
direction of top changes by $d\phi = \frac{\tau\, dt}{L}$

$$= \frac{l m g}{I\omega}\, dt$$

$$\dot{\phi} = \frac{mgl}{I\omega}$$

note $\omega \downarrow \Rightarrow \dot{\phi} \uparrow$

Now consider arbitrary tilt θ



$$\vec{L} = I\vec{\omega}$$

$$L_{\perp} = I\omega \sin\theta$$

$$L_{\parallel} = I\omega \cos\theta$$

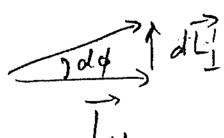
torque about pivot

$$\vec{\tau} = \vec{r} \times \vec{F}_g$$

$$|\tau| = l m g \sin\theta$$

Torque causes $\vec{L}_{\perp}$ to change direction, $\vec{L}_{\parallel}$ remains constant

From above



$$\frac{d\vec{L}_{\perp}}{dt} = \vec{\tau}$$

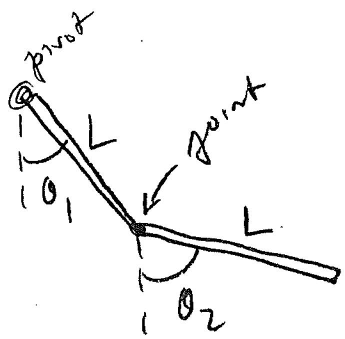
$$d\vec{L}_{\perp} = \vec{\tau}\, dt$$

$$d\phi = \frac{dL_{\perp}}{L_{\perp}} = \frac{|\tau|\, dt}{L_{\perp}} = \frac{l m g \sin\theta\, dt}{I\omega \sin\theta}$$

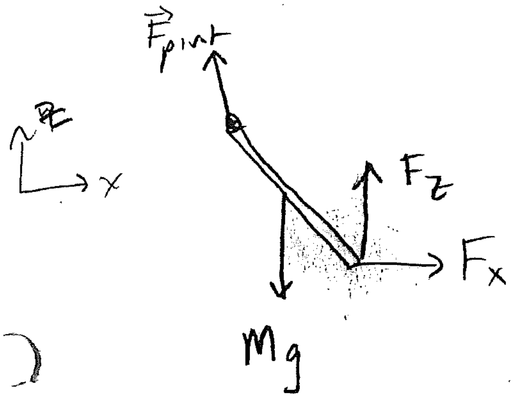
precession rate $\dot{\phi} = \dfrac{d\phi}{dt} = \dfrac{lmg}{I\omega}$ indep of angle of tilt

[ignore precession's contribution to $\vec{L}$ valid only in limit $\dot{\phi} \ll \omega$ ie $I\omega^2 \gg mgl$]

Double pendulum



Treat each limb as a system. Assume each limb has length L and mass M



Let F_x, F_z be components of force exerted on upper by lower limb

(• $\vec{F} = m\vec{a}$ not so useful because many unknown forces)

• Calc torque about pivot so we can ignore $\vec{F}_{pivot}$

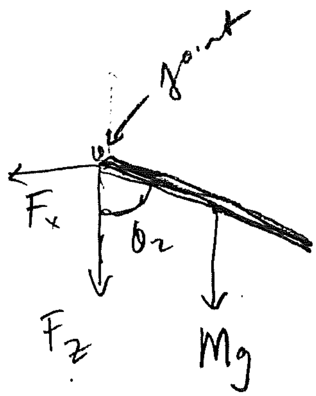
$$\tau_y = \frac{L}{2} Mg \sin\theta_1 - L F_x \cos\theta_1 - L F_z \sin\theta_1$$

$$= \frac{dL_y}{dt} = -I\ddot{\theta}_1 \qquad \text{where} \quad I = \frac{1}{3} ML^2$$

$$\Rightarrow \ddot{\theta}_1 = -\frac{3}{2}\frac{g}{L}\sin\theta_1 + \frac{3F_x}{ML}\cos\theta_1 + \frac{3F_z}{ML}\sin\theta_1$$

Define $\omega_0^2 \equiv \frac{g}{L}$

$$\boxed{\ddot{\theta}_1 = -\frac{3}{2}\omega_0^2 \sin\theta_1 + 3\frac{F_x}{ML}\cos\theta_1 + 3\frac{F_z}{ML}\sin\theta_1}$$



Forces on lower limb are equal + opposite

~~Calc torque about cm (accelerates)~~

→ can't calc about joint because it is not inertial.

But can use cm, even though accelerating

$$\tau_y = -\frac{L}{2} F_x \cos\theta_2 - \frac{L}{2} F_z \sin\theta_2$$

$$= \frac{dL_y}{dt} = -I' \ddot{\theta}_2 \quad \text{where} \quad I' = \frac{1}{12} ML^2$$

$$\Rightarrow \boxed{\ddot{\theta}_2 = \frac{6F_x}{ML} \cos\theta_2 + \frac{6F_z}{ML} \sin\theta_2}$$

2 eqns, 4 unknowns $\theta_1, \theta_2, F_x, F_z$

How determine F_x, F_z?

Use $\vec{F}^{ext} = m\vec{a}_{cm}$

Let x, z be cm of lower limb

$$-F_x = M\ddot{x}$$

$$-F_z - Mg = M\ddot{z}$$

$$x = L\sin\theta_1 + \frac{L}{2}\sin\theta_2$$

$$z = -L\cos\theta_1 - \frac{L}{2}\cos\theta_2$$

upper limb involves unknown pivot forces

Rest of calculat. for problem set

- Take derivative
- eliminate F_x + F_z from the 4 eqns
- write eqns for $\ddot{\theta}_1, \ddot{\theta}_2$

use force eqns

done in class

$$-F_x = M\ddot{x}$$

$$-F_z - Mg = M\ddot{z}$$

where x, z = c.m of lower limb.

$$x = L\sin\theta_1 + \frac{L}{2}\sin\theta_2 \qquad z = -L\cos\theta_1 - \frac{L}{2}\cos\theta_2$$

stop here

$$\dot{x} = L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2 \qquad \dot{z} = L\dot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_2\sin\theta_2$$

$$\ddot{x} = L(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\sin\theta_1 + \tfrac{1}{2}\ddot{\theta}_2\cos\theta_2 - \tfrac{1}{2}\dot{\theta}_2^2\sin\theta_2)$$

$$\ddot{z} = L(\ddot{\theta}_1\sin\theta_1 + \dot{\theta}_1^2\cos\theta_1 + \tfrac{1}{2}\ddot{\theta}_2\sin\theta_2 + \tfrac{1}{2}\dot{\theta}_2^2\sin\theta_2)$$

$$\Rightarrow \boxed{\begin{aligned} F_x &= mL\left(-\ddot{\theta}_1\cos\theta_1 + \dot{\theta}_1^2\sin\theta_1 - \tfrac{1}{2}\ddot{\theta}_2\cos\theta_2 + \tfrac{1}{2}\dot{\theta}_2^2\sin\theta_2\right) \\ F_z &= -Mg + mL\left(-\ddot{\theta}_1\sin\theta_1 - \dot{\theta}_1^2\cos\theta_1 - \tfrac{1}{2}\ddot{\theta}_2\sin\theta_2 - \tfrac{1}{2}\dot{\theta}_2^2\cos\theta_2\right) \end{aligned}}$$

Problem

P 5T

$$\ddot\theta_1 = -\frac{3}{2}\frac{g}{L}\sin\theta_1 + 3\left(-\ddot\theta_1\cos\theta_1 + \dot\theta_1^2\sin\theta_1 - \frac12\ddot\theta_2\cos\theta_2 + \frac12\dot\theta_2^2\sin\theta_2\right)\cos\theta_1$$

$$-3\frac{g}{L}\sin\theta_1 + 3\left(-\ddot\theta_1\sin\theta_1 - \dot\theta_1^2\cos\theta_1 - \frac12\ddot\theta_2\sin\theta_2 - \frac12\dot\theta_2^2\cos\theta_2\right)\sin\theta_1$$

$$\ddot\theta_1 = -\frac{9}{2}\frac{g}{L}\sin\theta_1 + 3\left[-\ddot\theta_1 - \frac12\ddot\theta_2\cos(\theta_2-\theta_1) + \frac12\dot\theta_2^2\sin(\theta_2-\theta_1)\right]$$

$$\ddot\theta_2 = 6\left(-\ddot\theta_1\cos\theta_1 + \dot\theta_1^2\sin\theta_1 - \frac12\ddot\theta_2\cos\theta_2 + \frac12\dot\theta_2^2\sin\theta_2\right)\cos\theta_2$$

$$-6\frac{g}{L}\sin\theta_2 + 6\left(-\ddot\theta_1\sin\theta_1 - \dot\theta_1^2\cos\theta_1 - \frac12\ddot\theta_2\sin\theta_2 - \frac12\dot\theta_2^2\cos\theta_2\right)\sin\theta_2$$

$$\ddot\theta_2 = -6\frac{g}{L}\sin\theta_2 + 6\left[-\ddot\theta_1\cos(\theta_1-\theta_2) + \dot\theta_1^2\sin(\theta_1-\theta_2) - \frac12\ddot\theta_2\right]$$

$$4\ddot\theta_1 + \frac32\ddot\theta_2\cos(\theta_1-\theta_2) = \frac32\dot\theta_2^2\sin(\theta_2-\theta_1) - \frac92\frac{g}{L}\sin\theta_1$$

$$4\ddot\theta_2 + 6\ddot\theta_1\cos(\theta_1-\theta_2) = 6\dot\theta_1^2\sin(\theta_1-\theta_2) - 6\frac{g}{L}\sin\theta_2$$

$$\boxed{\begin{aligned} 8\ddot\theta_1 + 3\ddot\theta_2\cos(\theta_1-\theta_2) &= 3\dot\theta_2^2\sin(\theta_2-\theta_1) - 9\omega_0^2\sin\theta_1 \\ 2\ddot\theta_2 + 3\ddot\theta_1\cos(\theta_1-\theta_2) &= 3\dot\theta_1^2\sin(\theta_1-\theta_2) - 3\omega_0^2\sin\theta_2 \end{aligned}}$$

$$8\ddot{\theta}_1 + 3\ddot{\theta}_2 = -9\omega_0^2\theta_1$$

$$2\ddot{\theta}_2 + 3\ddot{\theta}_1 = -3\omega_0^2\theta_2$$

$$6\ddot{\theta}_2 + 9\ddot{\theta}_1 = -9\omega_0^2\theta_2$$

$$-\ddot{\theta}_1$$

$$\theta_1 = e^{i\omega t}a_1$$

$$\theta_2 = e^{i\omega t}a_2$$

$$\begin{vmatrix} -8\omega^2 + 9\omega_0^2 & -3\omega^2 \\ -3\omega^2 & -2\omega^2 + 3\omega_0^2 \end{vmatrix} = 0$$

$$16\omega^4 - 42\omega^2\omega_0^2 + 27\omega_0^4 - 9\omega^4 = 0$$

$$7x^2 - 42x + 27 = 0 \qquad x = \frac{\omega^2}{\omega_0^2}$$

$$x = \frac{+42 \pm \sqrt{2^2\cdot 3^2\cdot 7^2 - 2^2\cdot 3^3\cdot 7}}{14}$$

$$= \frac{+42 \pm 2\cdot 3\sqrt{\underbrace{7^2 - 3\cdot 7}_{7\cdot 4}}}{14}$$

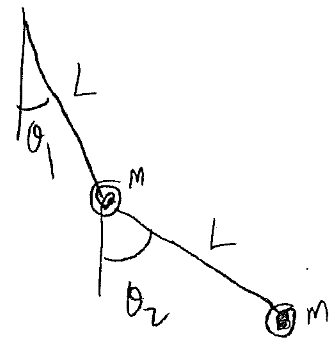
$$= +3 \pm \frac{6}{\sqrt{7}} = \begin{cases} 0.732213 \\ 5.26779 \end{cases}$$

ratio = 2.68?

$$\omega = \begin{cases} 0.8557\,\omega_0 \\ 2.295\,\omega_0 \end{cases}$$

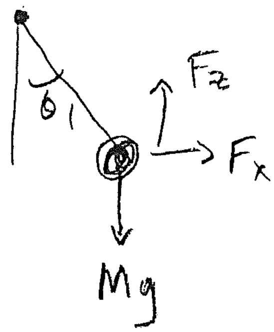
$$\omega_0 = \sqrt{\frac{g}{l}}$$

Double pendulum w/ massless rods!



The force exerted by the lower bob on the upper bob will be along the lower rod.

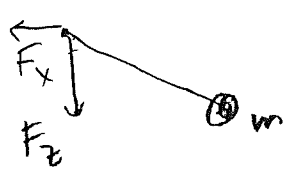
The torque about the cm of the lower bob must vanish because it has no moment of inertia



$$-I\ddot{\theta}_1 = \tau_y = LMg\sin\theta_1 - LF_x\cos\theta_1 - LF_z\sin\theta_1$$

$I = ML^2$ so letting $\omega_0^2 \equiv \frac{g}{L}$

$$\boxed{\ddot{\theta}_1 = -\omega_0^2\sin\theta_1 + \frac{F_x}{ML}\cos\theta_1 + \frac{F_z}{ML}\sin\theta_1}$$



$$\tau_y = -LF_x\cos\theta_2 - LF_z\sin\theta_2 = 0$$

$$\Rightarrow \boxed{F_x\cos\theta_2 + F_z\sin\theta_2 = 0}$$

$$-F_x = m\ddot{x}$$

$$-F_z - mg = m\ddot{z}$$

$$F_x = mL\left(-\ddot{\theta}_1\cos\theta_1 + \dot{\theta}_1^2\sin\theta_1 - \ddot{\theta}_2\cos\theta_2 + \dot{\theta}_2^2\sin\theta_2\right)$$

$$F_z = -mg + mL\left(-\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\cos\theta_1 - \ddot{\theta}_2\sin\theta_2 - \dot{\theta}_2^2\cos\theta_2\right)$$

Plug into prev eqns

$$\ddot{\theta}_1 = -2\omega_0^2\sin\theta_1 + \left(-\ddot{\theta}_1 - \ddot{\theta}_2\cos(\theta_2-\theta_1) + \dot{\theta}_2^2\sin(\theta_2-\theta_1)\right)$$

$$0 = -\omega_0^2\sin\theta_2 + \left(-\ddot{\theta}_1\cos(\theta_1-\theta_2) + \dot{\theta}_1^2\sin(\theta_1-\theta_2) - \ddot{\theta}_2\right)$$

$$2\ddot{\theta}_1 + \cos(\theta_2-\theta_1)\,\ddot{\theta}_2 = -2\omega_0^2\sin\theta_1 + \dot{\theta}_2^2\sin(\theta_2-\theta_1)$$

$$\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 + \ddot{\theta}_2 = -\omega_0^2\sin\theta_2 + \dot{\theta}_1^2\sin(\theta_1-\theta_2)$$

$$\begin{cases} 2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\omega_0^2\theta_1 \\ \ddot{\theta}_1 + \ddot{\theta}_2 = -\omega_0^2\theta_2 \end{cases}$$

$$\begin{vmatrix} +2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{vmatrix} = 0$$

$$\omega^4 - 4\omega^2\omega_0^2 + 2\omega_0^2 = 0 \qquad \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2$$