N0

Summary + review

$$\vec{p}^{\,sys} = M\vec{v}_{cm}$$

$$\frac{d\vec{p}^{\,sys}}{dt} = \vec{F}^{ext} \quad \text{in IRF}$$

$$\Rightarrow M\vec{a}_{cm} = \vec{F}^{ext}$$

If internal + external forces conservative (or constraint)

$$E = T^{sys} + \underbrace{U^{sys}}_{\text{ignore for rigid body}} + U^{ext} = \text{conserved}$$

$$T^{sys} = \tfrac{1}{2}Mv_{cm}^2 + T'^{\,sys}$$

For rigid body $T'^{\,sys} = \tfrac{1}{2}I'\omega^2$

$$T^{sys} = \tfrac{1}{2}(M r_{cm\perp}^2 + I')\omega^2 = \tfrac{1}{2}I\omega^2$$

$$\vec{L}^{sys} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{L}'^{\,sys}$$

For rigid body balanced about an axis: $\vec{L}'^{\,sys} = I'\vec{\omega}$

For origin at point on axis nearest cm

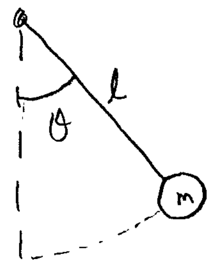
$$\vec{L}^{sys} = (M r_{cm\perp}^2 + I')\vec{\omega} = I\vec{\omega}$$

$$\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext} \quad \text{in IRF} \qquad \left(\text{also } \frac{d\vec{L}'^{\,sys}}{dt} = \vec{\tau}^{ext} \text{ in cm frame}\right)$$

$$\Rightarrow I\vec{\alpha} = \vec{\tau}^{ext}$$

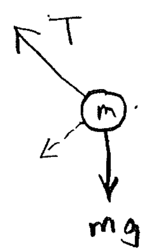
$\boxed{\text{Simple pendulum}}$: bob on a string

[done 3 ways!]



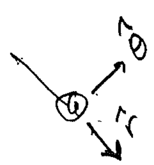
Force approach: $\vec{F} = m\vec{a}$

FBD



Tension directed along string.

We don't know its magnitude a priori so consider components of force in tangential direction



$$F_\theta = -mg\sin\theta$$
$$ma_\theta = ml\ddot{\theta}$$

assume equality of grav. and inertial mass

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Small angle $\sin\theta \sim \theta$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

ω_0 = angular frequency for small oscillations

Harmonic oscillator w/ $\omega_0 = \sqrt{\frac{g}{l}}$

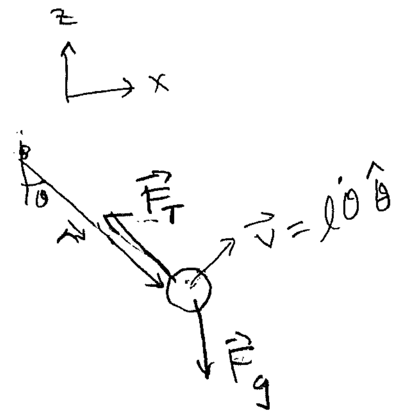
indep of mass!

Newton 1687 cited experiments he did to establish $m_{grav} = m_{iner}$.

Demonstrate equality of inertial + gravitational mass

N 2

Simple pendulum



Torque approach $\vec{\tau} = \dfrac{d\vec{L}}{dt}$

Calculate torque w/ origin at pivot.

$$\vec{r} \times \vec{F}_T = 0$$

$$\vec{r} \times \vec{F}_{grav} = l m g \sin\theta \,\hat{y}$$

Angular mom. of bob w/ orig at pivot

$$\vec{L} = \vec{r} \times m\vec{v} = m l (l\dot{\theta})(-\hat{y}) = -m l^2 \dot{\theta}\,\hat{y}$$

$$\frac{dL_y}{dt} = \tau_y$$

$$-m l^2 \ddot{\theta} = l m g \sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$\boxed{\text{Simple pendulum}}$

gravity (constraint forces do no work)

Energy approach: $T + U_{grav} = \text{const}$

$$\vec{v} = l\dot{\theta}\hat{\theta}$$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = -mgl\cos\theta$$

$\left\{ \begin{array}{l} \text{Initial angle } \theta_0 \\ \text{initial speed} = 0 \end{array} \right.$

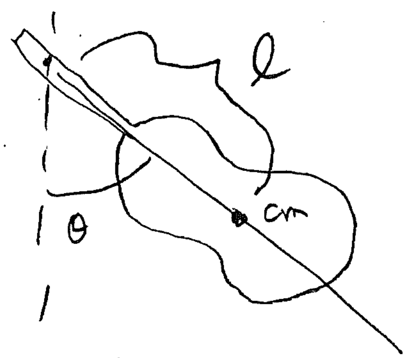
$$\frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0$$

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$$

differentiate to get $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

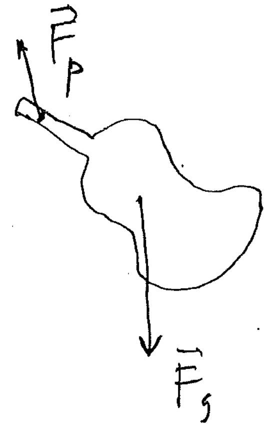
// compound pendulum (means distributed mass)

Physical pendulum : rigid object pivoting about one point



[internal forces cancel; need only consider external]

Force approach: FBD



$\vec{F}^{ext} = m\vec{a}_{cm}$

(2 eqns)

[don't dwell on location of force vectors until we get to torques]

We know neither the magnitude (a priori) nor the direction of $\vec{F}_p$

($\vec{F}_p$ not necessarily parallel to the line connecting pivot and cm)

Since F_{px} and F_{pz} unknown, force approach fails

[might think $\vec{F}_p$ is along line from pivot to cm but it is not

(only valid if radius of gyration is equal to distance to cm)

NJ
rev

Torque approach

choose origin at pivot

Torque due to pivot

$$\vec{\tau}_p = \vec{r} \times \vec{F}_p = 0 \qquad \text{because } \vec{r} = 0$$

[need neither F_{px} nor F_{py}]

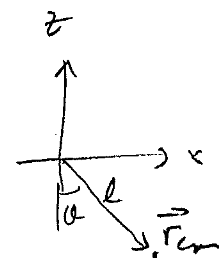
Torque due to gravity

$$\vec{\tau}_g = \int \vec{r} \times dm\,(-g\hat{z})$$

$$= g\hat{z} \times \underbrace{\int \vec{r}\, dm}_{\vec{r}_{cm}\underbrace{\int dm}_{m} + \underbrace{\int \vec{r}\,'\, dm}_{0}}$$

$$= mg\,\hat{z} \times \vec{r}_{cm}$$

$$= mg\ell \sin\theta\, \hat{y}$$



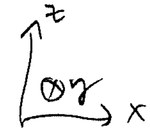
Gravitational torque acts as if all mass were concentrated at cm ← only because $\vec{g}$ is spatially uniform

("center of gravity")

object dynamically balanced wrt. $\vec{\omega}$ axis

$\vec{L} = I\vec{\omega}$ $\qquad I$ = mom of inertia wrt. pivot

$\vec{\omega} = -\dot{\theta}\,\hat{y}$



$L_y = I\omega_y = -I\dot{\theta}$

$$\frac{dL_y}{dt} = -I\ddot{\theta} = \tau_y = l m g \sin\theta$$

$$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0 \quad \overset{\text{small }\theta}{\Longrightarrow} \quad \omega_0^2 = \frac{mgl}{I} = \frac{g}{\left(\frac{I}{ml}\right)}$$

$\Rightarrow$ same period as simple pendulum of length $k = \frac{I}{ml}$

[French; Fetter & Walecka]

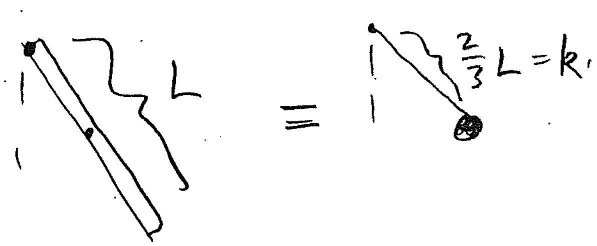
called "radius of gyration" of the physical pendulum

(same as center of percussion: see later)

Uniform rod of length $L \Rightarrow I = \frac{1}{3} M L^2 =$

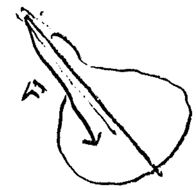
$l = \frac{L}{2}$

$$k = \frac{\frac{1}{3} m L^2}{\frac{mL}{2}} = \frac{2}{3} L$$



[not $\frac{L}{2}$]

Physical pendulum: energy approach



$$E = T^{sys} + U^{ext} = \text{const}$$

$$T^{sys} = \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot{\theta}^2$$

$$U^{ext} = \int dm\, g h = g \int dm\, \hat{z} \cdot \vec{r}$$

$$= g \hat{z} \cdot \int dm \left(\vec{r}_{cm} + \vec{r}\,' \right)$$

$$= g \underbrace{\hat{z} \cdot \vec{r}_{cm}}_{h_{cm}} \underbrace{\int dm}_{m} + g \hat{z} \underbrace{\int dm\, \vec{r}\,'}_{0,\ \text{trivial integral}}$$

$$= m g h_{cm}$$ so center of gravity = center of mass

$$= -m g l \cos\theta$$

$$E = \frac{1}{2} I \dot{\theta}^2 - m g l \cos\theta = -m g l \cos\theta_0$$

$$\dot{\theta}^2 = \frac{2 m g l}{I} \left(\cos\theta - \cos\theta_0 \right)$$

↓ differentiate

$$\ddot{\theta} + \frac{m g l}{I} \sin\theta = 0$$ as before

Problem

Calculate force exerted by pivot on a physical pendulum w/ mass m and moment of inertia I with respect to the pivot.

From class we know:
$$\begin{cases} \frac{1}{2} I \dot\theta^2 = mg\ell(\cos\theta - \cos\theta_0) \\ \ddot\theta = -\frac{mg\ell}{I}\sin\theta \end{cases}$$

$$x_{cm} = \ell\sin\theta$$
$$\dot x_{cm} = \ell\dot\theta\cos\theta$$
$$\ddot x_{cm} = \ell\ddot\theta\cos\theta - \ell\dot\theta^2\sin\theta$$

$$z_{cm} = -\ell\cos\theta$$
$$\dot z_{cm} = \ell\dot\theta\sin\theta$$
$$\ddot z_{cm} = \ell\ddot\theta\sin\theta + \ell\dot\theta^2\cos\theta$$

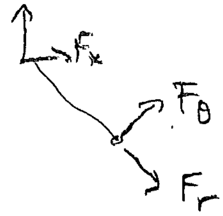
$$F_x^{pivot} = m\ddot x_{cm} = m\ell(\ddot\theta\cos\theta - \ell\dot\theta^2\sin\theta)$$
$$= m\ell\left(-\frac{mg\ell}{I}\sin\theta\cos\theta - \ell\frac{2mg\ell}{I}(\cos\theta - \cos\theta_0)\sin\theta\right)$$
$$= \frac{m^2 g\ell^2}{I}\sin\theta\left[-3\cos\theta + 2\cos\theta_0\right] < 0 \quad (\text{for } \theta > 0)$$

$$F_z^{pivot} = mg + m\ddot z_{cm}$$
$$= m[g + \ell(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)]$$
$$= m[g + \frac{mg\ell^2}{I}(\underbrace{-\sin^2\theta}_{\cos^2\theta - 1} + 2(\cos\theta - \cos\theta_0)\cos\theta)]$$
$$= \frac{m^2 g\ell^2}{I}\left[\left(\frac{I}{m\ell^2} - 1\right) - \cos\theta\left[-3\cos\theta + 2\cos\theta_0\right]\right)$$

If $I = m\ell^2$ (ie bob on a weightless rod)

then $\frac{F_x}{F_z} = -\tan\theta$ ie force is directed along rod

otherwise it is more vertical, ie $F_z > -F_x\tan\theta$



Alternatively,

$$F_x^{pivot} = F_r \sin\theta + F_\theta \cos\theta$$

$$F_z^{pivot} - mg = -F_r \cos\theta + F_\theta \sin\theta$$

But $$F_r = -m\dot{\theta}^2 l = \frac{2m^2 g l^2}{I}(\cos\theta - \cos\theta_0)$$

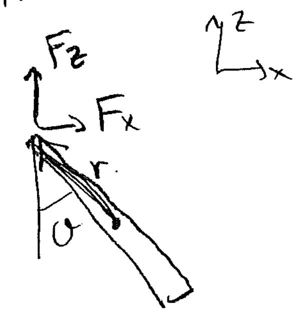
$$F_\theta = m\ddot{\theta} l = -\frac{m^2 g l^2}{I}\sin\theta$$

$$F_x^{pivot} = -m\dot{\theta}^2 l \sin\theta + m l \ddot{\theta}\cos\theta$$

$$F_z^{pivot} = mg + m\dot{\theta}^2 l \cos\theta + m l \ddot{\theta}\sin\theta$$

(cont.)

Now calculate the torque along the CM of the pendulum



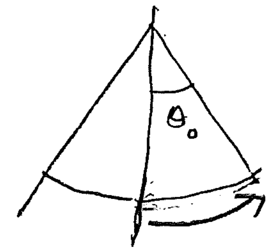
$$\tau_y = l F_x^{pivot} \cos\theta + l F_y^{pivot} \sin\theta$$

$$= \frac{m^2 g l^3}{I} \underbrace{\left(\frac{I}{ml^2} - 1 \right)}_{\frac{I'}{ml^2}} \sin\theta$$

$$= I' (- \ddot{\theta})$$

$$\Rightarrow \quad \ddot{\theta} + \frac{mgl}{I'} \sin\theta = 0$$

Period of physical pendulum



$$\left[E = \frac{1}{2} I \dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0\right]$$

$$\dot{\theta} = \pm\sqrt{\frac{2mgl}{I}(\cos\theta - \cos\theta_0)}$$

$$dt = \pm \frac{d\theta}{\sqrt{\frac{2mgl}{I}(\cos\theta - \cos\theta_0)}}$$

+ when swinging right
− when swinging left

During the upswing from 0 to θ_0

$$\int_0^{\frac{T}{4}} dt = \sqrt{\frac{I}{mgl}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}$$

Recall for small amplitude oscillations $\omega_0^2 = \frac{mgl}{I}$

$$\boxed{T = \frac{4}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}}$$

Phys. 3000: approx $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$$\Rightarrow T = \frac{4}{\omega_0} \underbrace{\int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}}}_{\frac{\pi}{2}} = \frac{2\pi}{\omega_0} \checkmark$$

Recall $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2}$

$$T = \frac{4}{\omega_0}\int_0^{\theta_0} \frac{d\theta}{2\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}$$

Let $\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{\theta_0}{2}\right)\sin\alpha \quad \Rightarrow \quad \alpha : 0 \to \frac{\pi}{2}$

$$\cos\left(\frac{\theta}{2}\right)\frac{d\theta}{2} = \sin\left(\frac{\theta_0}{2}\right)\cos\alpha\, d\alpha$$

$$T = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} \frac{\dfrac{2\sin\left(\frac{\theta_0}{2}\right)\cos\alpha\, d\alpha}{\cos\left(\frac{\theta}{2}\right)}}{2\sin\frac{\theta_0}{2}\sqrt{1-\sin^2\alpha}}$$

$$= \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\cos\frac{\theta}{2}}$$

$$= \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-\left(\sin^2\frac{\theta_0}{2}\right)\sin^2\alpha}} \quad \leftarrow \text{elliptic integral of 1st kind}$$

For θ small: $\int_0^{\frac{\pi}{2}} d\alpha = \frac{\pi}{2}$

Expand in θ_0

$$T = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} d\alpha\left[1 + \frac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2\alpha + \dots\right]$$

$$= \frac{4}{\omega_0}\left[\frac{\pi}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \dots\right] = \frac{2\pi}{\omega_0}\left[1 + \frac{\sin^2\frac{\theta_0}{2}}{4} + \dots\right]$$

$\theta_0 = 60 \Rightarrow \frac{1}{16}$

$\theta_0 = 90 \Rightarrow \frac{1}{8}$

$\theta_0 = 180 \Rightarrow$ should be ∞

If $\theta_0 = 180$, $\int \frac{d\alpha}{\sqrt{1-\sin^2\alpha}} = \int \frac{d\alpha}{\cos\alpha} = \int \sec\alpha\, d\alpha = \ln(\sec\alpha + \tan\alpha)\Big|_0^{\pi/2} = \infty$

N 10

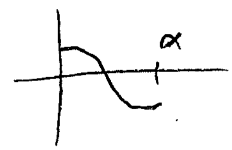
$$T = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \frac{K\left(\frac{k^2}{k^2-1}\right)}{\sqrt{1-k^2}}, \quad k = \sin\frac{\theta_0}{2}$$

Expand in powers of k.

$$(1-k^2\sin^2\alpha)^{-\frac{1}{2}} = 1 + \frac{k^2}{2}\sin^2\alpha + \frac{3k^4}{8}\sin^4\alpha$$

$$= 1 + \frac{k^2}{2}\left(\frac{1-\cos 2\alpha}{2}\right) + \frac{3k^4}{8}\left(\frac{1-2\cos 2\alpha+\cos^2 2\alpha}{4}\right)$$

Now $\displaystyle\int_0^{\frac{\pi}{2}} d\alpha \cos 2\alpha = \left.\frac{\sin 2\alpha}{2}\right|_0^{\frac{\pi}{2}} = 0$



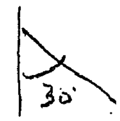
so $\cos 2\alpha$ terms drop out

$$T = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} d\alpha\left[1 + \frac{k^2}{4} + \frac{3k^4}{8}\left(\frac{1}{4} + \frac{1}{4}\underbrace{\cos^2 2\alpha}_{\text{averages to } \frac{1}{2}}\right) + \dots\right]$$

$$= \frac{4\; 2\pi}{\omega_0}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]$$

$$= \frac{2\pi}{\omega_0}\underbrace{\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \dots\right]}_{\text{corrections to small amplitude period}}$$

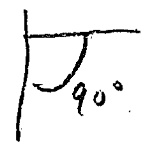
Let $\theta_0 = 30^\circ \Rightarrow k = 0.26$



$$T = \frac{2\pi}{\omega_0}\left[\underbrace{1 + 0.0167 + 0.0006}_{1.0173} + \dots\right]$$

full eval $= 1.0174$

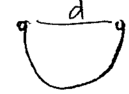
Let $\theta_0 = 90^\circ \Rightarrow k = 0.71$



$$T = \frac{2\pi}{\omega_0}\left[\underbrace{1 + 0.125 + 0.035}_{1.16} + \dots\right]$$

full $1.18034 = K($

$K\left(\frac{1}{2}\right) = \frac{\pi}{2}(1.18034) = 1.854$



$= \frac{1.18\pi}{\omega_0}$

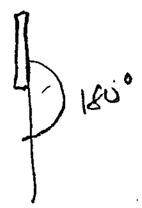
$= 1.18\pi\sqrt{\frac{\ell}{g}}$

$= \frac{1.18\pi}{\sqrt{2}}\sqrt{\frac{d}{g}}$

2.621

later compare w/ brachisto chron

Let $\theta_0 = 180^\circ$ $k = 1$



$$T = \frac{2\pi}{\omega_0}\left[\underbrace{1 + 0.25 + 0.14}_{1.39} + \dots\right.$$

Full $= \infty$!

$$\int \frac{d\alpha}{\sqrt{1-\sin^2\alpha}} = \int \sec\alpha \, d\alpha = \int \sec\alpha\left(\frac{\sec\alpha + \tan\alpha}{\sec\alpha + \tan\alpha}\right) d\alpha$$

$$= \ln\left|\sec\alpha + \tan\alpha\right|\Big|_0^{\frac{\pi}{2}} = \ln\left|\frac{\infty}{1}\right| = \infty.$$

Replace string & bob w/ frictionless wire

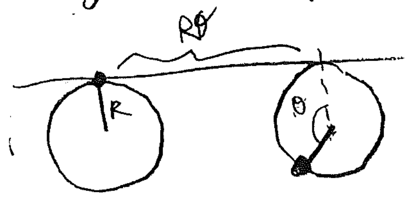
Circle $x = R \sin\theta$

$z = R(1-\cos\theta)$

Huygens: what curve gives amplitude-independent period?

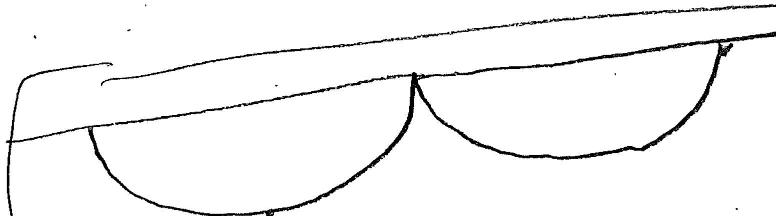
(tautochrone)

answer: cycloid (point on edge of circle rolling along a line



$x = R(\theta - \sin\theta)$

$z = -R(1-\cos\theta)$



expand around here

$\alpha = \pi + \beta$

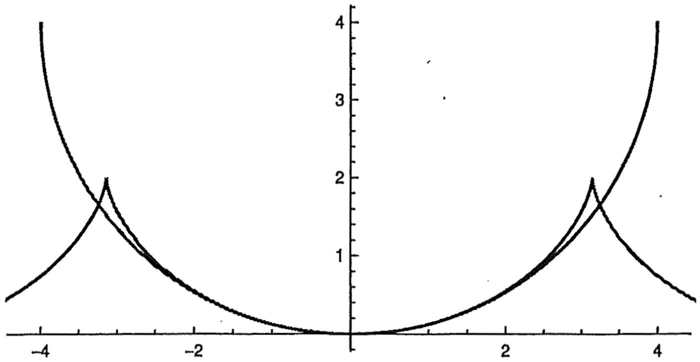
$x = R(\beta + \sin\beta)$

$z = R(1-\cos\beta)$

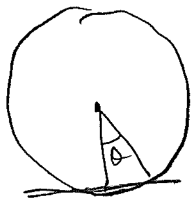
cycloidal pendulum

→ see figure in French...

```
Show[ ParametricPlot[ {4 Sin[a], 4 (1 - Cos[a])}, {a, -Pi / 2, Pi / 2}],
 ParametricPlot[ {a + Sin[a], 1 - Cos[a]}, {a, -3 Pi, 3 Pi}]]
```



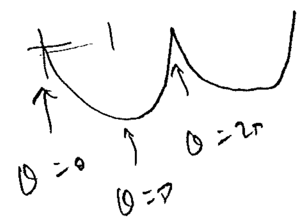
circle



$$x = R\sin\theta \approx R\theta$$
$$y = R(1-\cos\theta) \approx \tfrac{1}{2}R\theta^2 = \frac{1}{2R}x^2$$

so curvature = inverse of radius of curvature

cycloid



$$x = R(\theta - \sin\theta)$$
$$y = -R(1-\cos\theta)$$

Expand around $\theta = \pi + \alpha$

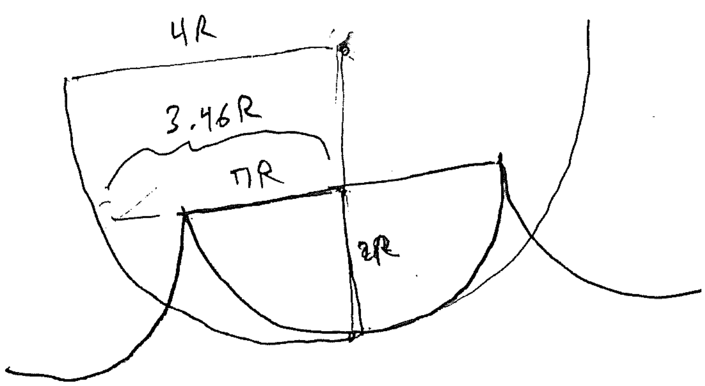
$\sin(\pi+\alpha) = \cos\pi \sin\alpha$
$\cos(\pi+\alpha) = \cos\pi\cos\alpha$

$$x = R\pi + R(\alpha + \sin\alpha) = R\pi + 2R\alpha$$
$$y = -R(1+\cos\alpha) = -2R + \tfrac{1}{2}R\alpha^2$$

Near bottom $\quad x' = 2R\alpha$
$$y' = \tfrac{1}{2}R\alpha^2 = \tfrac{1}{2}R\left(\frac{x'}{2R}\right)^2 = \frac{1}{8R}x'^2$$

$\Rightarrow$ radius of curvature of cycloid near bottom $= \underline{\underline{4R}}$

use this to do plot



1.732
2 3.4

$$(2R)^2 + y^2 = (4R)^2$$
$$y^2 = 12R^2 \qquad y = 2\sqrt{3}R \approx 3.46$$