Angular momentum of a system of particles

$$\vec{L}^{sys} = \sum \vec{L}_i = \sum m_i \vec{r}_i \times \vec{v}_i$$

in general $\vec{L}$ depends on choice of origin

Recall $\vec{r}_i{}' = \vec{r}_i - \vec{r}_{cm}$ = position relative to center-of-mass

$\sum m_i \vec{r}_i{}' = 0$ ie $\vec{r}_{cm}{}' = 0$

$\sum m_i \vec{v}_i{}' = 0$ ie $\vec{v}_{cm}{}' = 0$

Then

$$\vec{L}^{sys} = \sum m_i (\vec{r}_{cm} + \vec{r}_i{}') \times (\vec{v}_{cm} + \vec{v}_i{}')$$

$$= \underbrace{\left(\sum m_i\right)}_{M} \vec{r}_{cm} \times \vec{v}_{cm} + \underbrace{\left(\sum m_i \vec{r}_i{}'\right)}_{0} \times \vec{v}_{cm} + \vec{r}_{cm} \times \underbrace{\left(\sum m_i \vec{v}_i{}'\right)}_{0} + \underbrace{\sum m_i \vec{r}_i{}' \times \vec{v}_i{}'}_{\vec{L}_i{}'}$$

$$\vec{L}^{sys} = \underbrace{M \vec{r}_{cm} \times \vec{v}_{cm}}_{\text{angular momentum of system treated as a point particle}} + \underbrace{\vec{L}'^{\,sys}}_{\text{angular momentum measured in cm frame}}$$

$\vec{L}^{sys}$ does not depend on choice of origin iff $\vec{v}_{cm} = 0$.

Summary

$$\vec{p}^{\,sys} = M\,\vec{V}_{cm}$$ ← [no "internal" momentum]

$$T^{sys} = \frac{1}{2} M \vec{V}_{cm}^{\,2} + T'^{\,sys}$$

$$\vec{L}^{sys} = M\,\vec{r}_{cm} \times \vec{V}_{cm} + \vec{L}'^{\,sys}$$

} because cross terms cancel!

[not necessarily expected since T + L are both quadratic however due to nature of cm frame cross terms cancel]

change in angular momentum of a system

$$\vec{L}^{sys} = \sum \vec{r}_i \times \vec{p}_i$$

$$\frac{d\vec{L}^{sys}}{dt} = \sum \Big[\underbrace{\cancel{\frac{d\vec{r}_i}{dt} \times \vec{p}_i}}_{0 \text{ since } \vec{p}_i \parallel \vec{v}_i} + \vec{r}_i \times \underbrace{\frac{d\vec{p}_i}{dt}}_{\vec{F}_i} \Big]$$

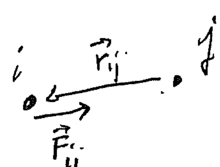
in IRF

$$= \sum_i \vec{r}_i \times \Big[\vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \Big]$$

$$= \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{ext}}_{\text{define this as } \vec{\tau}^{ext}} + \underbrace{\sum_i \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}}$$

$$\sum_{\text{pairs } i<j} \underbrace{\Big[\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij}} \Big]}_{(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}}$$

A central force is one in which $\vec{F}_{ij}$ is parallel (or antiparallel) to $\vec{r}_{ij}$. [eg gravity, electrostatic]



Thus $\vec{r}_{ij} \times \vec{F}_{ij} = 0$ [internal torques cancel]

For central internal forces we have:

$$\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext} \quad \text{in an IRF}$$

[analogous to $\frac{d\vec{p}^{sys}}{dt} = \vec{F}^{ext}$]

Corollary: $\vec{L}^{sys}$ is conserved if external torque vanishes
[also holds for individual components]

K4

Observe that $\vec{\tau}^{ext} = \sum \vec{r}_i \times \vec{F}_i^{ext}$

in general $\vec{\tau}$ depends on choice of origin

$$= \sum_i (\vec{r}_{cm} + \vec{r}_i{}') \times \vec{F}_i^{ext}$$

$$= \vec{r}_{cm} \times \underbrace{\sum \vec{F}_i^{ext}}_{\vec{F}^{ext}} + \underbrace{\sum \vec{r}_i{}' \times \vec{F}_i^{ext}}_{}$$

$\vec{\tau}'^{\,ext}$ = ext. torque measured wrt. origin of cm

---

Recall that $\vec{L}^{sys} = M\vec{r}_{cm} \times \dot{\vec{r}}_{cm} + \vec{L}'^{\,sys}$

$$\frac{d\vec{L}^{sys}}{dt} = \underbrace{M\vec{r}_{cm} \times \ddot{\vec{r}}_{cm}}_{\vec{r}_{cm} \times \vec{F}^{ext}} + \frac{d\vec{L}'^{\,sys}}{dt}$$

Comparing previous eqns:

$$\boxed{\frac{d\vec{L}'^{\,sys}}{dt} = \vec{\tau}'^{\,ext}}$$

change in angular mom. <u>wrt. cm</u>
equals external torque <u>wrt. cm</u>

→ valid even if cm frame is <u>not inertial!</u>

[cf Fetter & Walecka]

summary

$$\frac{d\vec{p}^{\,sys}}{dt} = \vec{F}^{ext} \quad \text{(valid in IRF)}$$

(ditto)

$$\frac{dE^{sys}_{mech}}{dt} = \frac{dW^{ext}}{dt} = P^{ext}$$ ← power delivered by external force

$$\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext}$$ ← torque caused by external force

(valid in IRF $\underline{\underline{\cap}}$ in cm frame)

Rigid body

a system of particles whose relative positions are fixed modulo an overall translation + rotation (shape cannot change)

In particular, all $|\vec{r}_{ij}|$ are const.

[eg. point masses connected by weightless rods (chemistry models of molecules)]

[atoms in a solid at their equilibrium positions]

[talk informally about d.o.f. 3N for N-pt masses
but only 6 for a rigid body $\Rightarrow$ euler angles]

Recall

$$E_{mech}^{sys} = T^{sys} + U^{sys}$$

because internal forces are conservative

$$dE_{mech}^{sys} = dW^{ext}$$

If external forces are conservative $dW^{ext} = -dU^{ext}$

$$d(E_{mech}^{sys} + U^{ext}) = 0$$

$$E^{tot} = E_{mech}^{sys} + \underbrace{U^{sys}}_{U^{internal}} + U^{ext}$$ is conserved

$$U^{internal} = \sum_{i<j} U_{ij}(|\vec{r}_{ij}|)$$

For particles at equilibrium positions, $U^{internal} = \text{const}$ and can be ignored

(internal constraint forces do no work)

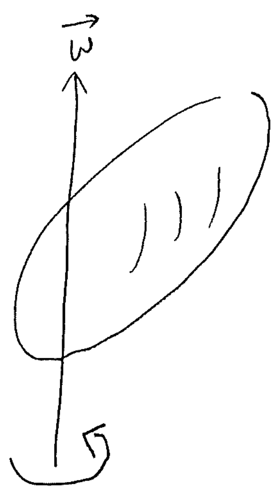
Rigid body $\Rightarrow$ $E^{tot} = T^{sys} + U^{ext}$

Rotating rigid body

Assume rigid body is rotating about some fixed axis

[not necessarily thru cm]

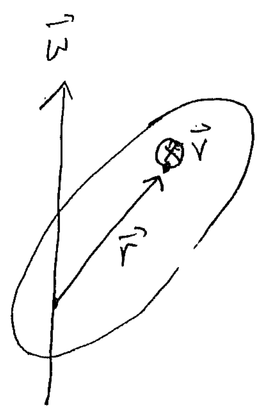
$\Rightarrow$ angular velocity $\vec{\omega}$



$\hat{\omega}$ = axis of rotation [r.h. rule]

$\omega = |\vec{\omega}|$ = angular speed

[need not be constant]



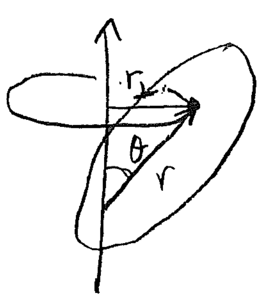
Let $\vec{r}$ denote a point within the object (w.r.t. origin on the axis)

The velocity of this point is

$$\vec{v} = \vec{\omega} \times \vec{r}$$

check direction ✓

check magnitude



$$|\vec{v}| = \omega r \sin\theta = \omega r_\perp \quad ✓$$

$r_\perp$ = perpendicular distance from axis

Kinetic energy of rotating object

$$T^{sys} = \int \frac{1}{2} (dm)\, v^2$$

$$= \frac{1}{2} \omega^2 \int dm\, r_\perp^2$$

$$= \frac{1}{2} I \omega^2$$

dx dy dz = dV

mass density

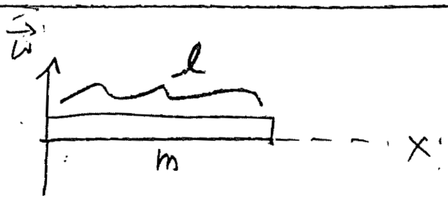
Moment of inertia

$$I = \int dm\, r_\perp^2 = \int d^3r\, \rho\, r_\perp^2$$

(depends on the axis!)

---

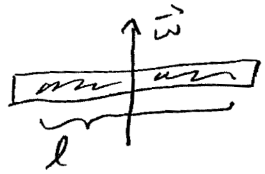
eg. riged rod about one end:



Linear density $\lambda = \frac{m}{l}$

$$I = \int \lambda\, dx\, r_\perp^2 = \frac{m}{l} \int_0^l dx\, x^2 = \frac{1}{3} m l^2$$

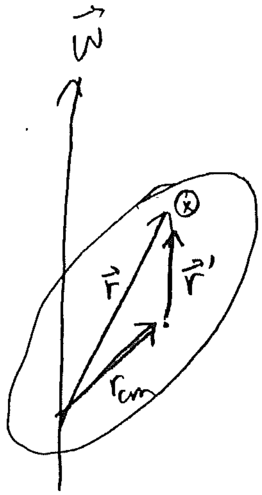
Riged rod about center



$= 2$

$$I = 2\, \frac{1}{3} \left(\frac{m}{2}\right)\left(\frac{l}{2}\right)^2 = \frac{1}{12} m l^2$$

L5



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$T^{sys} = \frac{1}{2}\int dm\, (\vec{\omega} \times \vec{r})^2$$

$$\vec{r} = \vec{r}_{cm} + \vec{r}\,'$$

$$\vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{r}_{cm} + \vec{\omega} \times \vec{r}\,'$$

$$T^{sys} = \frac{1}{2}\int dm (\vec{\omega} \times \vec{r}_{cm})^2 + \int dm\, (\vec{\omega} \times \vec{r}_{cm}) \cdot (\vec{\omega} \times \vec{r}\,') + \frac{1}{2}\int dm\, (\vec{\omega} \times \vec{r}\,')^2$$

$$= \frac{1}{2}\underbrace{(\vec{\omega} \times \vec{r}_{cm})^2}_{\omega^2 r^2_{cm\perp}} \underbrace{\int dm}_{M} + (\vec{\omega} \times \vec{r}_{cm}) \cdot \Big(\vec{\omega} \times \underbrace{\int dm\, \vec{r}\,'}_{0}\Big) + T'^{\,sys}$$

Kinetic energy in cm frame

$$T^{sys} = \frac{1}{2} M v_{cm}^2 + T'^{\,sys}$$

[confirming earlier result]

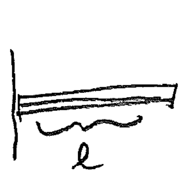
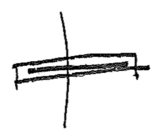
Note: cm frame is moving + accelerating but not rotating so object is rotating in cm frame at same angular speed ω as in the rest frame

$$\frac{1}{2} I \omega^2 = \frac{1}{2} M r^2_{cm\perp} \omega^2 + \frac{1}{2} I' \omega^2$$

mom of inertia about axis thru cm

$$\boxed{I = M r^2_{cm\perp} + I'}$$

parallel axis theorem

check:  $= M\left(\frac{l}{2}\right)^2 +$ 

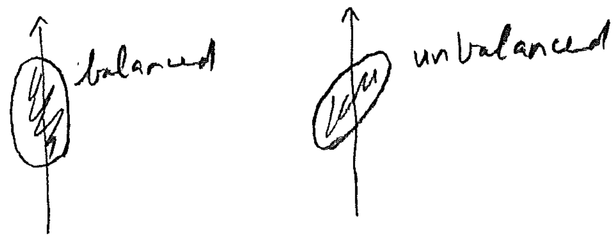
$$\frac{1}{3} m l^2 = \frac{1}{4} m l^2 + \frac{1}{12} m l^2$$

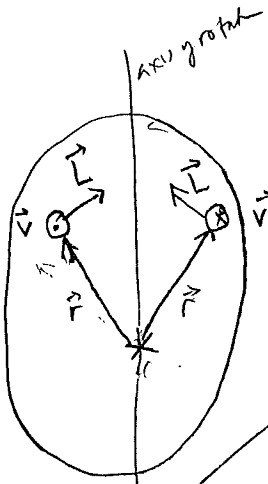
[problems]

Balanced objects

For every element on one side of axis
there is an equal element on the opposite side

$\Rightarrow$ axis goes through cm

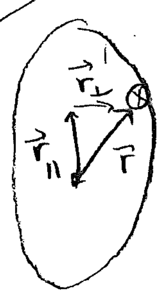




$$\vec{L} = \int dm \;\; \vec{r} \times \vec{v}$$

[can choose origin at cm, but need not; must be on axis, though]

Radial components of $\vec{L}$ cancel
leaving only a component along $\vec{\omega}$



$$\vec{r} \times \vec{v} = \underbrace{\vec{r}_{\parallel} \times \vec{v}}_{\text{radial}} + \underbrace{\vec{r}_\perp \times \vec{v}}_{}$$

$$(r_\perp v)\, \hat{\omega}$$

$$= r_\perp^2 \omega \, \hat{\omega}$$

$$= r_\perp^2 \, \vec{\omega}$$

$$\vec{L} = \int dm \, \vec{r} \times \vec{v} = \int dm \, r_\perp^2 \vec{\omega} = I \vec{\omega}$$

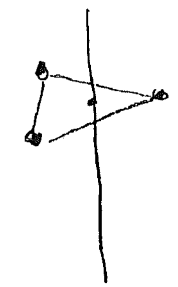
For a balanced object, $\vec{L}$ is parallel to $\vec{\omega}$
(not for unbalanced)

$$\left[\vec{r} \times \vec{v} = \vec{r} \times (\vec{\omega} \times \vec{r}) = r^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r} \right]$$

[Slater + Frank, p.99 Mechanics = on dynamical vs static balance]

More generally, we call an object dynamically balanced w.r.t. axis $\hat{\omega}$ if

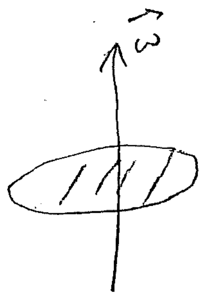
$\vec{L}$ is parallel to $\vec{\omega}$.



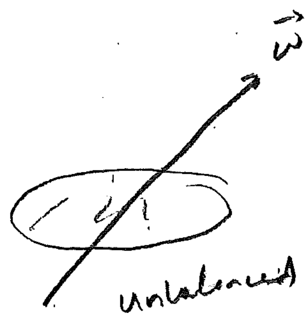
← dynamically balanced

(not balanced in earlier sense).

Being "balanced" depends not only on object but also on the axis



balanced



unbalanced

We will later show that for any object (however misshapen)

there are at least 3 (perpendicular) axes

about which it is dynamically balanced (called principle axes)

Recall from earlier, for a general system

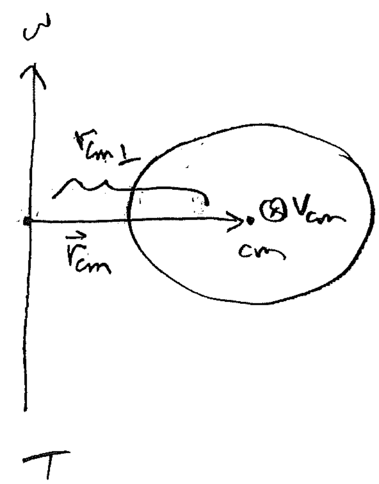
$$\vec{L}^{sys} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{L}'^{sys}$$

ang. mom w.r.t. cm

For a balanced object rotating about axis through cm

$$\vec{L}'^{sys} = I'\vec{\omega}$$

Consider axis <u>not</u> passing through cm [∴ not balanced]



Choose origin for $\vec{L}$ to be the point on the axis <u>closest</u> to cm

Then $\vec{r}_{cm} \times \vec{v}_{cm} = r_{cm\perp} v_{cm} \hat{\omega}$

$$= r_{cm\perp} (r_{cm\perp}\omega) \hat{\omega}$$

$$= r_{cm\perp}^2 \vec{\omega}$$

$$\vec{L}^{sys} = M r_{cm\perp}^2 \vec{\omega} + I'\vec{\omega}$$

$$= I\vec{\omega}$$ by parallel axis theorem

So $\vec{L} = I\vec{\omega}$ valid for any balanced object provided origin is at point on axis closest to cm

For an object dynamically balanced wrt axis $\hat{\omega}$

$$\vec{L} = I\vec{\omega}$$

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$ $\vec{\alpha}$ = angular acceleration

Recall $\frac{d\vec{L}}{dt} = \vec{\tau}$ valid in any IRF
and also in cm frame even if accelerating

$$\vec{\tau} = I\vec{\alpha}$$

analog of

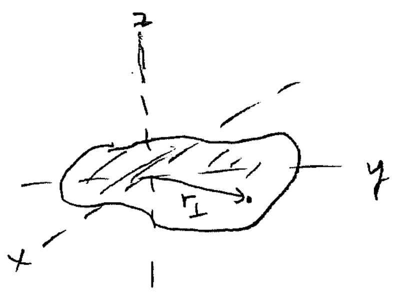
$$\vec{F} = m\vec{a}$$

Perpendicular axis theorem

- Applies only to <u>flat</u> objects
  (assume object lies in $z=0$ plane)

Let I_{zz} = moment of inertia wrt. z-axis. [explain double index later

$$= \int dm\, r_\perp^2 = \int dm\,(x^2+y^2)$$



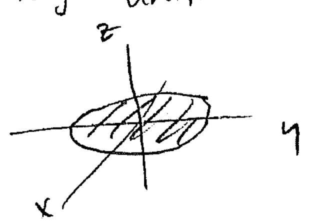
$$I_{xx} = \int dm\,(y^2 + \cancel{z^2}^{\,0})$$

$$I_{yy} = \int dm\,(x^2 + \cancel{z^2}^{\,0})$$

$$\Rightarrow \quad I_{zz} = I_{xx} + I_{yy}$$

---

e.g. uniform disk has $I_{zz} = \frac{1}{2} m R^2$



$I_{xx} = I_{yy}$ by symmetry

$$\Rightarrow \quad I_{xx} = \frac{1}{4} m R^2$$

A solid object can often be built from a stack of flat objects