Virial theorem

Define the virial of a system of self-interacting particle

$$G = \sum_i \vec{r}_i \cdot \vec{p}_i$$

Consider the change of virial

$$\frac{dG}{dt} = \sum \vec{v}_i \cdot \vec{p}_i + \sum \vec{r}_i \cdot \frac{d\vec{p}_i}{dt}$$

$$= \sum \frac{\vec{p}_i}{m} \cdot \vec{p}_i + \sum \vec{r}_i \cdot \vec{F}_i$$

$$= \sum \frac{p_i^2}{m} + \sum \vec{r}_i \cdot \sum_{j \neq i} \vec{F}_{ij}$$

$$= 2T^{sys} + \sum_{i<j} (\vec{r}_i \cdot \vec{F}_{ij} + r_j \cdot \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij}})$$

$$= 2T^{sys} + \sum_{i<j} (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{ij}$$

N.B. contact forces don't contribute because only act when $\vec{r}_i - \vec{r}_j = 0$.

Suppose the particle interact via ~~an attractive~~ a conservative force

$$U_{ij} = \frac{k_{ij}}{n} r_{ij}^n \quad , \left(k_{ij} > 0 \text{ for attractive}\right)$$

e.g. $n = 2$ $\qquad U = \frac{1}{2} k r^2$ $\qquad$ (harmonic osc.)

$n = 1$ $\qquad U = -\frac{k}{r}$ $\qquad$ (gravity, Coulomb)

$$\vec{F}_{ij} = -\vec{\nabla} U_{ij} = -\frac{\partial U_{ij}}{\partial r_{ij}} \hat{r}_{ij} = -k_{ij} r_{ij}^{n-1} \hat{r}_{ij}$$

Then

$$\begin{aligned}
\sum_{i<j} \vec{r}_{ij} \cdot \vec{F}_{ij} &= -\sum k_{ij} r_{ij}^{n-1} \hat{r}_{ij} \cdot \vec{r}_{ij} \\
&= -\sum k_{ij} r_{ij}^n \\
&= -n \sum_{i<j} U_{ij} \\
&= -n \, U^{sys}
\end{aligned}$$

$$\Rightarrow \frac{dG}{dt} = 2T^{sys} - n \, U^{sys}$$

$$\int_0^{\tau} \frac{dG}{dt} = \int_0^{\tau} (2T - nU)\,dt$$

$$G(\tau) - G(0) = \tau\,(2\langle T\rangle - n\langle U\rangle)$$

time averages

$$\frac{G(\tau) - G(0)}{\tau} = 2\langle T\rangle - n\langle U\rangle$$

If system is confined to a finite region ($|\vec{r}_i| < R$)

then $G = \sum \vec{r}_i \cdot \vec{p}_i$ is bounded

Let $\tau \to \infty$, then l.h.s $\to 0$

$$\Rightarrow \quad \boxed{2\langle T^{sys}\rangle = n\langle U^{sys}\rangle}$$

eg $n=2$ (harmonic osc) $\quad \langle T^{sys}\rangle = \langle U^{sys}\rangle$

$n=-1$ (grav, or Coulomb) $\quad \langle T^{sys}\rangle = -\frac{1}{2}\langle U^{sys}\rangle$

or $E^{sys} = \langle T^{sys}\rangle + \langle U^{sys}\rangle = -\langle T^{sys}\rangle$

eg Bohr model

(a) $Q = \frac{4\pi}{3}\rho R^3$

$$Q(r) = Q\frac{r^3}{R^3} = \frac{4\pi}{3}\rho r^3$$

$$dq(r) = \frac{3Qr^2 dr}{R^3} = 4\pi r^2 \rho\, dr$$

$$dW = \frac{KQ(r)\,dq}{r} = \frac{3KQ^2 r^4}{R^6}dr = \frac{(4\pi)^2 K\rho^2 r^4\,dr}{3}$$

$$W = \frac{3KQ^2}{5R} = \frac{(4\pi)^2 K\rho^2 R^5}{15}$$

(b) $U^{sys} = -\frac{3}{5}\frac{Gm^2}{R}$

$$E_{mech} = -\frac{3}{10}\frac{Gm^2}{R} = -\frac{3}{10}\frac{(6.67E-11)(2E30)^2}{(7E8\text{ m})} = -1.14E41\text{ J}$$

$$L_\odot = 3.8E26\text{ W} \Rightarrow t = \frac{E_{mech}}{L_\odot} = 3E14\text{ s} = 10\text{ million yrs.}$$

(c) $\langle K\rangle = -\frac{1}{2}\langle U\rangle = \frac{3}{10}\frac{G(Nm)^2}{R} = \frac{1}{2}(Nm)\langle v^2\rangle$

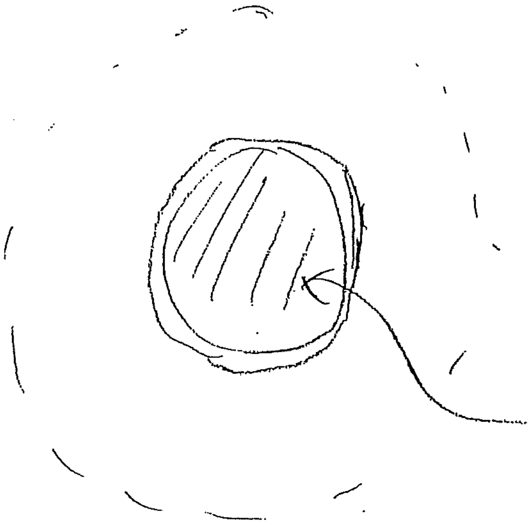
$$\boxed{\langle v^2\rangle = \frac{3}{5}\frac{G(Nm)}{R}}$$

$$M = Nm = \frac{5}{3}\frac{R\langle v^2\rangle}{G} = \frac{5}{3}\frac{(10^{22}\text{ m})(1.5E6\text{ m/s})^2}{(6.67E-11)} = 5.6E44\text{ kg}$$

$$M_{Lum} = (800)(10^9)(2E30) = 1.6E42\text{ kg}$$

Ratio = 350

$E = T + U = \frac{1}{2}U$



$\vec{F}\cdot d\vec{r} = $ 8π

$\rho = \frac{3M}{4\pi R^3}$

$m(r) = \frac{4\pi r^3}{3}\rho \qquad dm = 4\pi r^2 \rho\, dr$

$= M\frac{r^3}{R^3} \qquad = 3M\frac{r^2 dr}{R^3}$

so $\boxed{U = G\int_0^r \frac{m(r)\,dr}{r} = \frac{3GM^2}{\cancel{4}R^6}\int_0^r r^4 dr = \frac{3}{5}G\frac{M^2}{R}}$

Coma: $R \approx 10^{22}$ m

$M \approx (800 \times 10^9) \times 2 \times 10^{30}$ kg $= \underline{1.6 \times 10^{42}\text{ kg}}$

$\bar{v} \approx 1{,}500\ \frac{\text{km}}{\text{s}} \Rightarrow \bar{v}^2 = 2.25\times10^{12}\ \text{m}^2/\text{s}^2$

$\bar{K} = -\frac{1}{2}\bar{U} = \frac{1}{2}\frac{3}{5}\frac{GM^2}{R}$

$\Rightarrow \cancel{M}\bar{v}^2 = \frac{3}{5}\frac{GM^{\cancel{2}}}{R} \Rightarrow M \approx \frac{5}{3}\frac{R\bar{v}^2}{G}$

$= \frac{5}{3}\frac{10^{22}\text{ m} \times 2.25\times10^{12}\text{ m}^2/\text{s}^2}{6.67\times10^{-11}\text{ m}^3\text{ s}^{-2}}$ kg

$= \underline{5.6\times10^{44}\text{ kg}}$

Gravity

$$2\left\langle \sum \frac{1}{2} m_i v_i^2 \right\rangle = -\left\langle -\sum_{i<j} \frac{G m_i m_j}{r_{ij}} \right\rangle$$

$$\left\langle \sum m_i v_i^2 \right\rangle = \frac{1}{2}\left\langle \sum_{i\neq j} \frac{G m_i m_j}{r_{ij}} \right\rangle$$

assume N equal masses m

$$N m \langle v^2 \rangle = \frac{G N^2 m^2}{2} \left\langle \frac{1}{r} \right\rangle$$

$$\langle v^2 \rangle = \frac{G N m}{2} \left\langle \frac{1}{r} \right\rangle$$

But $M_{tot} \approx Nm$

$$\langle v^2 \rangle = \frac{G M_{tot}}{2} \left\langle \frac{1}{r} \right\rangle$$

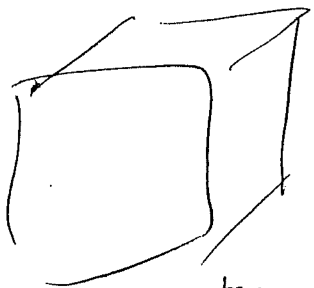
measure avg $\langle v^2 \rangle$ & avg $\langle \frac{1}{r} \rangle$ to find $\underline{M_{tot}}$.

Alternatively: $\underbrace{2\langle T \rangle}_{(\sum m_i)\langle v^2 \rangle} = -\langle U \rangle = +\frac{3}{5}\frac{GM^2}{R}$

$$\langle v^2 \rangle = \frac{3}{5}\frac{GM}{R}$$

~~K 10~~
1994 notes

Derivation of ideal gas law



ideal, monatomic

what are forces?

force of constraint.

collision forces cancel out ~~because particles~~

$\cancel{\vec F_i \cdot \vec r_i}$

$\vec F_{ij} = -\vec F_{ji}$

because particles are at same place $\vec r_i = \vec r_j$

i.e. $\underbrace{(\vec r_i - \vec r_j)}_{0 \text{ when } \vec F_{ij} \neq 0} \cdot \vec F_{ij} = 0$

$$\underbrace{\langle T \rangle}_{\frac{3}{2}NkT} = -\frac{1}{2}\left\langle \sum \vec F_i \cdot \vec r_i \right\rangle$$

$\underbrace{\sum \vec F_i}_{-P\,d\vec A}$ since $d\vec A \sim \hat n$

[Force imparted by wall per unit area $= P$

$P\,dA\,dt = \sum F_i\,dt$]

$$= \frac{1}{2} P \underbrace{\int d\vec A \cdot \vec r}_{\int \underbrace{\nabla \cdot \vec r}_{3}\, dV}$$

$$= \frac{3}{2} PV$$

$PV = NkT$

[what about diatomic $\frac{5}{2}kT$, $\frac{7}{2}kT$]

(combining T & U)

Rotational coordinates not bounded since no restoring force ...