## Kinetic energy of a system of particles

$$T^{sys} = \sum_{i=1}^{N} T_i = \sum \tfrac{1}{2} m_i \vec{v}_i^{\,2}$$

Recall $\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$

where $\vec{v}_{cm} = \frac{1}{M} \sum m_i \vec{v}_i$

and $\vec{v}_i'$ = velocity relative to cm velocity

= velocity measured in cm frame [possibly noninertial]

Recall $\sum m_i \vec{v}_i' = 0$ (total momentum = 0 in cm frame)

$$T^{sys} = \sum \tfrac{1}{2} m_i (\vec{v}_{cm} + \vec{v}_i') \cdot (\vec{v}_{cm} + \vec{v}_i')$$

$$= \tfrac{1}{2} \left(\sum m_i\right) v_{cm}^2 + \sum m_i \vec{v}_i' \cdot \vec{v}_{cm} + \sum \tfrac{1}{2} m_i \vec{v}_i'^{\,2}$$

$$= \tfrac{1}{2} M v_{cm}^2 + \underbrace{\left(\sum m_i \vec{v}_i'\right)}_{0} \cdot \vec{v}_{cm} + T'^{sys}$$

$$T^{sys} = \underbrace{\tfrac{1}{2} M v_{cm}^2}_{\substack{\text{Kin en of} \\ \text{particle of mass } M \\ \text{+ speed } v_{cm}}} + \underbrace{T'^{sys}}_{\substack{\text{"internal kinetic energy"} \\ \text{(measured in cm frame)}}}$$

or eg thermal energy

Work-energy theorem for system of particles

$$dT_i = \vec{F}_i \cdot d\vec{r}_i \qquad \text{(work done on particle } i\text{)}$$

$$= \vec{F}_i^{\,ext} \cdot d\vec{r}_i + \sum_{j \neq i} \vec{F}_{ij}(\vec{r}_{ij}) \cdot d\vec{r}_i \qquad \text{where } \vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$$

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$$dT^{sys} = \sum dT_i$$

$$= \underbrace{\sum_i \vec{F}_i^{\,ext} \cdot d\vec{r}_i}_{dW^{ext}} + \underbrace{\sum_i \left[ \sum_{j>i} \vec{F}_{ij}(\vec{r}_{ij}) \cdot d\vec{r}_i + \sum_{j<i} \vec{F}_{ij}(\vec{r}_{ij}) \cdot d\vec{r}_i \right]}$$

(work done by external force on system)

$$\underbrace{\sum_{\substack{\text{pairs} \\ i<j}} \left[ \vec{F}_{ij} \cdot d\vec{r}_i + \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij} \text{ by 3rd law}} \cdot d\vec{r}_j \right]}$$

$$\sum_{i<j} \vec{F}_{ij} \cdot \underbrace{(d\vec{r}_i - d\vec{r}_j)}_{d\vec{r}_{ij}}$$

If forces are conservative, then

$$\vec{F}_{ij} = -\vec{\nabla} U_{ij}(\vec{r}_{ij}) \Rightarrow \vec{F}_{ij} \cdot d\vec{r}_{ij} = -dU_{ij}$$

( If forces are constraint forces, then

$$\vec{F}_{ij} \perp d\vec{r}_{ij} \Rightarrow \vec{F}_{ij} \cdot d\vec{r}_{ij} = 0$$ )

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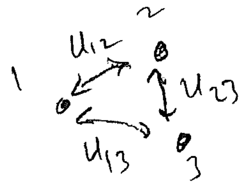
$$\sum_i dT_i + \sum_{\substack{\text{pairs} \\ i<j}} dU_{ij} = dW^{ext}$$




Mechanical energy of system

$$E^{sys}_{mech} = \sum_i T_i + \sum_{i<j} U_{ij} = T^{sys} + U^{sys}$$

$$dE^{sys}_{mech} = dW^{ext}$$

(all internal work is encoded in internal potential energy $U^{sys}$)

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Isolated system (no external forces)
$\Rightarrow$ conservation of system's mechanical energy

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Also $T^{sys} = \frac{1}{2} M v_{cm}^2 + T'^{sys}$ so

$$E^{sys}_{mech} = \underbrace{\frac{1}{2} M v_{cm}^2}_{} + \underbrace{T'^{sys} + U^{sys}}_{\text{mechanical energy in cm frame}}$$

separately conserved for isolated system

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Note: particles also have rest energy $m_i c^2$ [e.g. radioactive decay]
but unless particles' identities change
these are conserved, so can ignore

Unnecessary: we do this later in n3



Two body system

$$m_1\vec{v}_1' + m_2\vec{v}_2' = 0$$

$$\vec{v}_2' = -\frac{m_1}{m_2}\vec{v}_1'$$

$$\vec{v}_1' - \vec{v}_2' = \left(1+\frac{m_1}{m_2}\right)\vec{v}_1' = \frac{M}{m_2}\vec{v}_1'$$

$$\Rightarrow \vec{v}_1' = \frac{m_2}{M}(\vec{v}_1' - \vec{v}_2')$$

$$\vec{v}_2' = \frac{m_1}{M}(\vec{v}_1' - \vec{v}_2')$$

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$$T'^{sys'} = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right)(\vec{v}_1' - \vec{v}_2')^2$$

$$= \frac{1}{2}\mu(\vec{v}_1' - \vec{v}_2')^2 = \frac{1}{2}\mu(\vec{v}_1 - \vec{v}_2)^2$$

Thus

$$T^{sys} = \frac{1}{2}M v_{cm}^2 + \frac{1}{2}\mu(\vec{v}_1 - \vec{v}_2')^2$$

confirming result found in problem

~~$E^{sys}_{meso} = T^{sys} + U^{sys}$~~

~~$E = T_{cm} + T'^{sys} + U^{sys}$ separately conserved for isolated sys~~

Lets include rest mass of constituents $m_i c^2$

$$E = \cancel{T_{cm}} + T'^{sys} + U + \sum m_i c^2$$

But for a bound state (eg proton = 3 quarks)

$$E = T_{cm} + Mc^2$$

Then

$$M = \sum m_i + \frac{T'^{sys} + U^{sys}}{c^2}$$

proton

1 GeV

$< 10$ MeV

probably best to avoid this

Well, U somewhat ill defined for microscopic system

also T as we've defined it not valid for relativistic constituents

$$U^{sys} = \sum_{\substack{\text{pairs}\\ i<j}} U_{ij}$$

$\longrightarrow$ do I2 first

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$U^{sys}$ can be understood as work required to assemble system from infinitely separated components

(assume $U_{ij}(r_{ij}) \to 0$ as $r_{ij} \to \infty$)

Begin w/ $m_1$ at $\vec{r}_1$ + all others at $\infty$ (and infinitely separated)

[$\infty$ is a big place!]

Hold $m_1$ fixed, bring $m_2$ from $\infty$ to $\vec{r}_2$.

If $\vec{F}_{21}$ repulsive, need exert inward force to bring in at const. speed, thus doing positive work on 2

$$W_2 = -\int_\infty^{r_2} \vec{F}_2 \cdot d\vec{r}_2 = -\int_\infty^{r_{21}} \vec{F}_{21} \cdot d\vec{r}_{21} = \int dU_{21} = U_{21}(r_{21}) - U_{21}(\infty)$$

No work done on $m_1$ since it's stationary.

Hold $m_1$ + $m_2$ fixed, bring in $m_3$

$$W_3 = -\int_\infty^{r_3} \vec{F}_3 \cdot d\vec{r}_3 = -\int \vec{F}_{31} \cdot d\vec{r}_3 - \int \vec{F}_{32} \cdot d\vec{r}_3$$

$$= U_{31}(r_{31}) + U_{32}(r_{32})$$

etc.

Total work done $= \sum_{i<j} U_{ij}(r_{ij})$

If force attractive, negative work done $\Rightarrow U^{sys} < 0$.

Coulomb energy of sphere of uniform charge density

$$U^{sys} = \sum_{i<j} \frac{K q_i q_j}{r_{ij}} = \frac{K}{2} \sum_i \sum_{j \neq i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$$\rightarrow \frac{K}{2} \iint \frac{\rho\, d^3\vec{r}\ \rho\, d^3\vec{r}'}{|\vec{r} - \vec{r}'|}$$

Rather than do this double integral, calculate work done to assemble sphere of charge $Q$ and radius $R$.

Problem

Assume have already assembled a sphere of radius $r$ having done work $U(r)$.

Assembled charge is $q(r) = Q \frac{r^3}{R^3} = \frac{4}{3}\pi \rho r^3$, $\rho = \frac{Q}{(\frac{4}{3}\pi R^3)}$

Bring in a shell of thickness $dr$ and charge $dq = \rho\, 4\pi r^2 dr = \frac{3Q}{R^3} r^2 dr$

Work done is $\frac{K q(r)\, dq}{r} = \frac{3KQ^2}{R^6} r^4 dr$

$$U(r + dr) - U(r) = \frac{3KQ^2}{R^6} r^4 dr$$

$$\frac{dU}{dr} = \frac{3KQ^2}{R^6} r^4$$

$$U = \frac{3KQ^2 r^5}{5R^6} \Big|_0^R = \frac{3KQ^2}{5R}$$

gravitational energy $U = -\frac{3GM^2}{5R}$