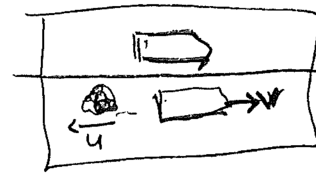
Rockets

In free space $\vec{F}^{ext} = 0 \Rightarrow \vec{p}^{sys}$ conserved

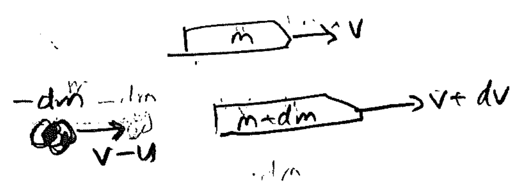


rocket frame

How does a rocket accelerate?

By expelling mass

[2022 Thomas did not do rockets]



change in rocket mass $dm < 0$

mass of exhaust $-dm > 0$

u = const speed w/ which exhaust expelled from rocket

e.g. chemical process (LOX/Kerosene) Saturn V $u \sim 2500\ m/s$

[for chemical rockets Moore says $u_{max} \sim 4500\ m/s$]

$$dp^{sys} = (m+dm)(v+dv) + (-dm)(v-u) \quad - mv$$

$$= mv + m\,dv + v\,dm + \underbrace{\cancel{dm\,dv}}_{\text{neglect}} - v\,dm + u\,dm - mv$$

$$dp^{sys} = m\,dv + u\,dm = 0$$

$$m\frac{dv}{dt} = \underbrace{u\left(-\frac{dm}{dt}\right)}_{\text{define the quantity as thrust}} > 0$$

$$ma = F_{thrust}$$

Even though system is isolated, expulsion of exhaust acts as an effective (thrust) force on the (remaining) mass

$$m\, dv = -u\, dm$$

$$\int dv = -u \int \frac{dm}{m}$$

$$v(t) - v_0 = -u\left[\ln m(t) - \ln m_0\right]$$

$$v(t) = v_0 + u \ln\left[\frac{m_0}{m(t)}\right]$$

Change in speed depends only on exhaust speed u and ratio of initial to final mass (not on $\frac{dm}{dt}$)

Most of initial mass is fuel

$$m_0 = M_{rocket} + m_{fuel}$$

structural constraints $\Rightarrow \frac{m_{fuel}}{m_{rocket}} \lesssim 10-20$, $\ln(20) \sim 3$

$u = 2500\ m/s$ $\Rightarrow v_{max} \lesssim 7500\ m/s$ [disregarding gravity]

but $v_{escape} \sim 11{,}000\ m/s$

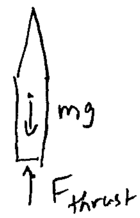
need multiple stages

Ascent under gravity

$$\frac{dp^{sys}}{dt} = F^{ext}$$

$$m\frac{dv}{dt} + u\frac{dm}{dt} = -mg$$

$$m\frac{dv}{dt} = u\left(-\frac{dm}{dt}\right) - mg$$



Assume mass is expelled at a constant rate

$$\frac{dm}{dt} = -R = \text{const} \quad \rightarrow F_{thrust} = uR$$

$$m\frac{dv}{dt} = uR - mg$$

Require $uR > mg$ for lift off [Now R matters]

$$\frac{dv}{dt} = \frac{uR}{m} - g$$

$$m = m_0 - Rt$$

$$\frac{dv}{dt} = \frac{uR}{(m_0 - Rt)} - g$$

acceleration increases as rocket loses mass.

[Problem] find v and z by integrating

$$v = v_0 - gt + u\log\left(\frac{m_0}{m_0 - Rt}\right)$$

$$z = z_0 + v_0 t - \frac{1}{2}gt^2 + ut - \frac{u}{R}(m_0 - Rt)\log\left(\frac{m_0}{m_0 - Rt}\right)$$

Saturn V rocket used for Apollo missions

- initial mass $m_0 = 2.8 \times 10^6$ kg [French p.327]
- initial weight $m_0 g = 2.74 \times 10^7$ N = 6.2 million lbs [1 lb = 4.45 N]
- exhaust speed $u = 2{,}500 \frac{m}{s}$

For lift-off need $R > \frac{m_0 g}{u} = 11{,}000 \frac{kg}{s}$

Saturn V had $R = 14{,}000 \frac{kg}{s}$ [15 tons/sec]

thrust $uR = 3.5 \times 10^7$ N = 7.9 million lbs.

- mass of first stage fuel $m_1 = 2.1 \times 10^6$ kg $= \frac{3}{4} m_0$
- time of first stage burn $t = \frac{m_1}{R} = \frac{2.1 \times 10^6 \text{ kg}}{14000 \text{ kg/s}} = 150$ sec $= 2\frac{1}{2}$ min

[French, p.327]

[show videos]

Problem: initial accel: $(2.5 \frac{m}{s^2})$
accel near end of 1st stage burn $(40 \frac{m}{s^2})$
speed at end of 1st stage burn $(2000 \frac{m}{s})$
height at end of 1st stage burn: vertical ascent (90 km)

Apollo solution

(a) initial acceleration: $a(0) = \frac{uR}{m_0} - g = \overbrace{\frac{(2500)(14000)}{(2.8E6)}}^{12.5\ m/s^2} - 9.8 = \boxed{2.7\ \frac{m}{s^2}}$

accel at end of 1st stage $a(150) = \frac{uR}{\left(\frac{m_0}{4}\right)} - g = 50 - 9.8 = \boxed{40.2\ \frac{m}{s^2}}$

(b) $dv = -g\,dt - u\frac{dm}{m}$ or $\frac{dv}{dt} = -g + \frac{uR}{m(t)}$

$v - v_0 = -gt - u\ln\left(\frac{m}{m_0}\right)$ $\quad = -g + \frac{uR}{m_0 - Rt}$

$v(t) = v_0 - gt + u\ln\left(\frac{m_0}{m(t)}\right)$ $\quad v = v_0 - gt + u\log\left(\frac{m_0}{m_0 - Rt}\right)$

velocity at end of 1st stage

$v(150) = v_0 - gt_1 + u\ln 4 = 0 - (9.8)(150s) + (2500)(1.386) = \boxed{1996\ \frac{m}{s}}$

(c) $dz = v_0\,dt - gt\,dt + u(\ln m_0)dt - u\underbrace{\left(\frac{dt}{dm}\right)}^{-\frac{1}{R}}\ln m\,dm$

$z = z_0 + v_0 t - \frac{1}{2}gt^2 + \underbrace{u(\ln m_0)t + \frac{u}{R}(m\ln m - m)\Big|_{m_0}^{m}}$

$\underbrace{u(\ln m_0)t + \frac{um\ln m}{R} - \frac{um}{R} - \frac{um_0}{R}\ln m_0 + \frac{um_0}{R}}$

$\frac{u\,m(t)}{R}\ln\left(\frac{m(t)}{m_0}\right) + ut$

$= z_0 + v_0 t + ut - \frac{1}{2}gt^2 + \frac{u}{R}(m_0 - Rt)\ln\left(\frac{m_0 - Rt}{m_0}\right)$

$v_0 = z_0 = 0 \Rightarrow \quad z = ut - \frac{1}{2}gt^2 - \frac{u\,m(t)}{R}\ln\left(\frac{m(t)}{m_0}\right)$

$= \underbrace{(2500)(150)}_{3.75E5} - \underbrace{\frac{1}{2}(9.8)(150)^2}_{1.10E5} - \underbrace{(2500)\frac{(7E5)}{(1.4E4)}\ln 4}_{1.73E5} = 9.1\times10^5\ m$

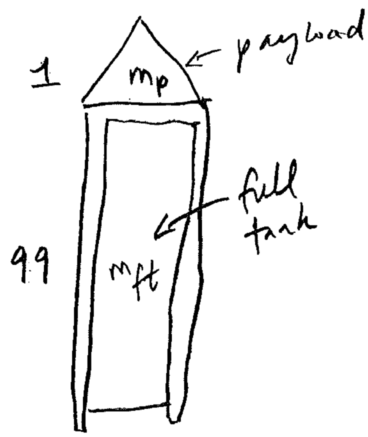
$= 91\ km$

GSI

• Max velocity for single stage rocket $\sim u \log\left(\frac{m_i}{m_f}\right)$

mass ratio is limited by structural constraints

Most of mass is fuel, but need container to hold it



Let $m_p = 1$, $m_{ft} = 99$

$$M_i = m_p + m_{ft} = 100$$

Suppose ratio of full tank to empty is 9:1

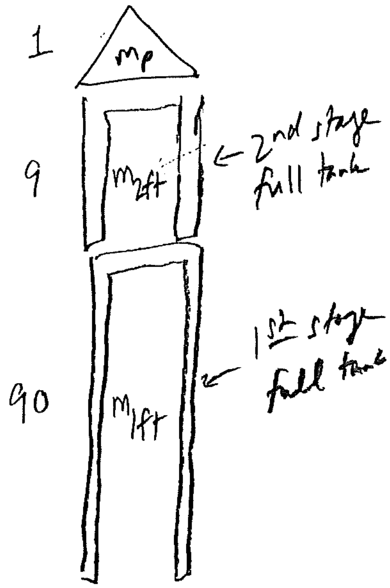
$$m_{et} = 11$$

$$M_f = m_p + m_{et} = 12$$

$$\log\left(\frac{M_i}{M_f}\right) = \log\left(\frac{100}{12}\right) \sim 2.1$$

Final velocity (ignore gravity) $v = v_0 + u \log\left(\frac{M_i}{M_f}\right) \sim v_0 + 2.1\,u$

Instead split fuel into two stages



$m_p = 1$

$m_{2ft} = 9 \qquad m_{2et} = 1$

$m_{1ft} = 90 \qquad m_{1et} = 10$

Initial rocket mass

$$M_i = m_p + m_{2ft} + m_{1ft} = 100$$

At end of first stage burn

$$M_{1f} = m_p + m_{2ft} + m_{1et} = 1 + 9 + 10 = 20$$

$$\log\left(\frac{M_i}{M_{1f}}\right) = \log\left(\frac{100}{20}\right) = 1.6$$

$$V_1 = V_0 + u \log\left(\frac{M_i}{M_{1f}}\right)$$

Now eject empty 1st stage tank

$$M_{2i} = m_p + m_{2ft} = 1 + 9 = 10$$

At end of 2nd stage burn

$$M_{2f} = m_p + m_{2et} = 1 + 1 = 2$$

$$\log\left(\frac{M_{2i}}{M_{2f}}\right) = \log\left(\frac{10}{2}\right) = 1.6$$

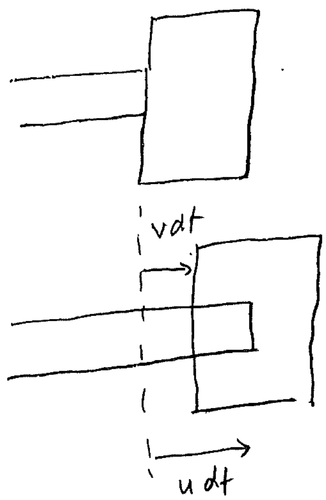
$$V_2 = V_1 + u \log\left(\frac{M_{2i}}{M_{2f}}\right)$$

$$V_2 = V_0 + u \log\left(\frac{M_{1i}}{M_{1f}}\right) + u \log\left(\frac{M_{2i}}{M_{2f}}\right)$$

$$= V_0 + 3.2\,u.$$

Hose and sponge problem

Jet has linear mass density λ



In time t, jet has moved $u\,dt$, sponge has moved $v\,dt$ so a length $dl = (u-v)\,dt$ has entered sponge and therefore

$$dm = \lambda\,dl = \lambda(u-v)\,dt \qquad (1)$$

Also
$$m v + dm\,u = (m+dm)(v+dv)$$
$$= mv + dm\,v + m\,dv$$

$$dm(u-v) = m\,dv$$

$$\frac{dm}{m} = \frac{dv}{u-v} \qquad (2)$$

$$\ln m \Big|_{m_0}^{m} = -\ln(u-v)\Big|_0^{v}$$

$$\ln\frac{m}{m_0} = -\ln\left(\frac{u-v}{u}\right)$$

$$m = \frac{m_0 u}{u-v} \qquad (3)$$

Several ways to proceed from here

Solve ③ for $u - v = \frac{m_0 u}{m}$

Sub into ① $dm = \lambda \frac{m_0 u}{m} dt$

$$m\,dm = \lambda m_0 u\, dt$$

$$\frac{m^2}{2} - \frac{m_0^2}{2} = \lambda m_0 u t$$

$$m = \sqrt{m_0^2 + 2\lambda m_0 u t}$$

Then $v = u - \frac{m_0 u}{m} = u\left[1 - \frac{m_0}{\sqrt{m_0^2 + 2\lambda m_0 u t}}\right] = u\left[1 - \frac{1}{\sqrt{1 + \frac{2\lambda u t}{m_0}}}\right]$

Alternatively ② $\Rightarrow dm = \frac{m\,dv}{u-v} \underset{③}{=} \frac{m_0 u}{(u-v)^2} dv$

but ① $\Rightarrow dm = \lambda (u-v)\, dt$ ~~so~~

$$\lambda\, dt = \frac{m_0 u}{(u-v)^3} dv$$

$$\lambda t = \frac{m_0 u}{2(u-v)^2} - \frac{m_0 u}{2u^2}$$

$$\frac{1}{(u-v)^2} = \frac{1}{u^2} + \frac{2\lambda t}{m_0 u} = \frac{1}{u^2}\left[1 + \frac{2\lambda t u}{m_0}\right]$$

$$u - v = \frac{u}{\sqrt{1 + \frac{2\lambda u t}{m_0}}}$$

$$v = u\left[1 - \frac{1}{\sqrt{1 + \frac{2\lambda u t}{m_0}}}\right] \checkmark$$

then ③ $\Rightarrow m = \frac{m_0 u}{u - v} = m_0\sqrt{1 + \frac{2\lambda u t}{m_0}} = \sqrt{m_0^2 + 2\lambda m_0 u t}$ ✓

Two-stage rocket problem

$m_{1i} = Nm$

$m_{1F} = r(N-n)m + nm = rNm + n(1-r)m$

$m_{2i} = nm$

$m_{2f} = r(n-1)m + m = (1-r)m + nrm$

(a)

$$\boxed{\begin{array}{l} \Delta V_1 = u \log\left(\frac{N}{rN + n(1-r)}\right) \\ \hline \Delta V_2 = u \log\left(\frac{n}{(1-r) + nr}\right) \end{array}}$$

$r=0 \Rightarrow \ln\left(\frac{N}{n}\right)$

$r=1 \Rightarrow \Delta V = 0$

$r=0 \Rightarrow u \ln(n)$

(b) $\frac{d}{dn}(\Delta V_1 + \Delta V_2) = 0$

$$-\frac{1-r}{rN + n(1-r)} + \frac{1}{n} - \frac{r}{(1-r)+nr} = 0$$

$$\frac{(1-r) + \cancel{nr} - \cancel{nr}}{n[(1-r)+nr]} = \frac{1-r}{rN + n(1-r)}$$

$$rN + \cancel{n(1-r)} = \cancel{n(1-r)} + n^2 r$$

$$n^2 = N$$

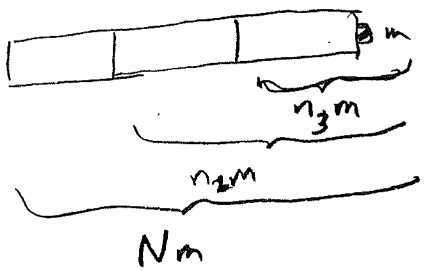
$$\boxed{n = \sqrt{N}}$$

(c) $\Delta V_1 = u \log \left(\frac{N}{rN + \sqrt{N}(1-r)} \right)$

$\Delta V_2 = u \log \left(\frac{\sqrt{N}}{(1-r) + \sqrt{N}\, r} \right)$

$$\boxed{\Delta V_1 = \Delta V_2}$$

(d)



$m_{2f} = (n_2 - n_3) rm + n_3 m \Rightarrow \begin{cases} \Delta V_3 = u \log \left(\frac{n_3}{n_3 r + (1-r)} \right) \\ \Delta V_2 = u \log \left(\frac{n_2}{n_2 r + n_3 (1-r)} \right) \\ \Delta V_1 = u \log \left(\frac{N}{N r + n_2 (1-r)} \right) \end{cases}$

If $n_3 = N^{1/3}$ and $n_2 = N^{2/3}$, the all these ΔV are equal $\Rightarrow$ $\Delta V = u \log \left(\frac{N^{1/3}}{N^{1/3} r + (1-r)} \right)$

(But haven't actually proved that this maximizes ΔV_{tot})

$\Delta V_{tot} = 3u \log \left(\frac{N^{1/3}}{N^{1/3} r + (1-r)} \right)$

p-stages: $\Delta V_{tot} = pu \log \left(\frac{N^{1/p}}{N^{1/p} r + (1-r)} \right)$