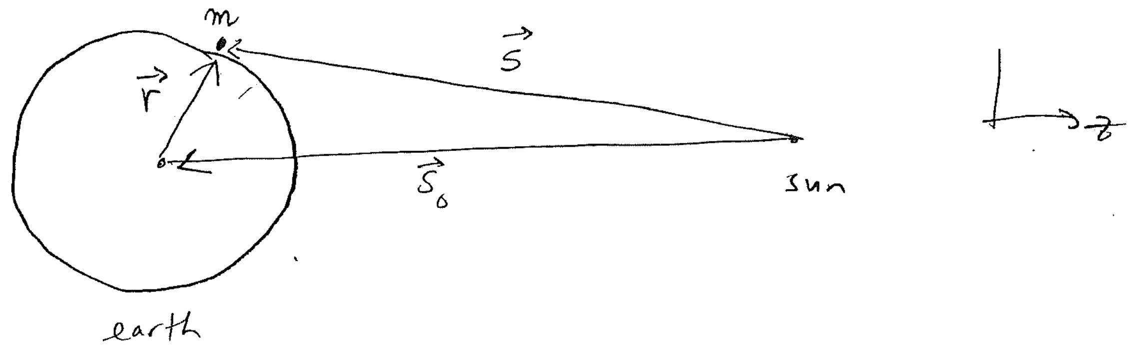
Tides

[cf John Taylor, p. 333]



Force of sun on a mass m near earths surface

$\vec{s}$ = radius vectr wrt sun

$$\vec{F}_s = -\frac{GM_s m}{s^2}\hat{s}$$

Earth is accelerating toward sun

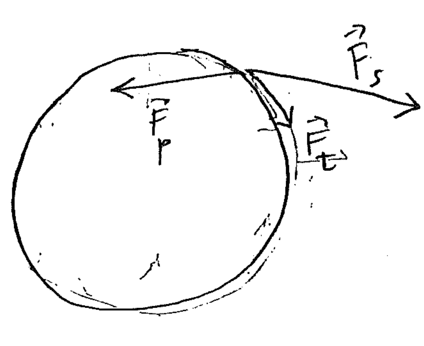
$$\vec{a}_0 = -\frac{GM_s}{s_0^2}\hat{s}_0 = \frac{GM_s}{s_0^2}\hat{z}$$

In earth's cm frame, there is a pseudo force on m

$$\vec{F}_p = -m\vec{a}_0 = -\frac{GM_s m}{s_0^2}\hat{z}$$

Tidal force is the net force on the mass

$$\vec{F}_t = \vec{F}_s + \vec{F}_p = -\frac{GM_s m}{s^2}\hat{s} - \frac{GM_s m}{s_0^2}\hat{z}$$



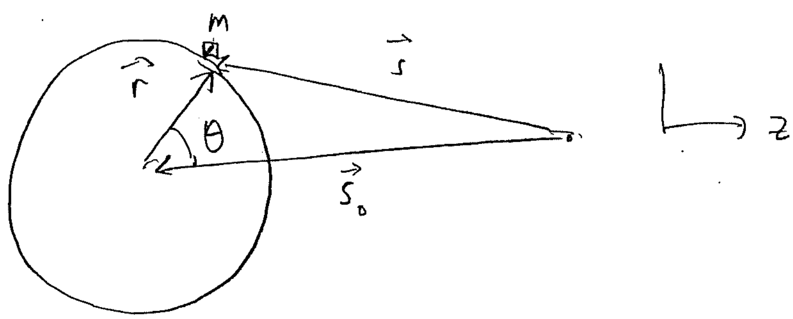
[show figure in French]
[will draw it later in notes]

The tidal force is conservative because can write

$$\vec{F}_t = -\vec{\nabla} U_t$$

$$U_t = -\frac{GM_s m}{s} + \frac{GM_s m}{s_0^2} z$$

$$= -\frac{GM_s m}{s_0}\left(\frac{s_0}{s} - \frac{z}{s_0}\right)$$



Let's write U_t in terms of θ

$$z = r\cos\theta$$

$$\vec{s} = \vec{s}_0 + \vec{r}$$

$$s^2 = s_0^2 + 2\vec{s}_0\cdot\vec{r} + r^2$$

$$= s_0^2 - 2s_0 r\cos\theta + r^2$$ [also law of cosines]

$$= s_0^2\left[1 - \frac{2r\cos\theta}{s_0} + \frac{r^2}{s_0^2}\right].$$

Recall $(1+\epsilon)^{\alpha} = 1 + \alpha\epsilon + \frac{\alpha(\alpha-1)}{2}\epsilon^2$

$$\frac{s_0}{s} = \left(1 - \frac{2r\cos\theta}{s_0} + \frac{r^2}{s_0^2}\right)^{-1/2}$$

$$= 1 - \frac{1}{2}\left(-\frac{2r\cos\theta}{s_0} + \frac{r^2}{s_0^2}\right) + \frac{3}{8}\left(-\frac{2r\cos\theta}{s_0} + \frac{r^2}{s_0^2}\right)^2 + \ldots$$

$$= \underbrace{1}_{P_0(\cos\theta)} + \frac{r}{s_0}\underbrace{\cos\theta}_{P_1(\cos\theta)} + \frac{r^2}{s_0^2}\underbrace{\left(\frac{3}{2}\cos^2\theta + \frac{1}{2}\right)}_{P_2(\cos\theta)} + \ldots$$

Recall legendre poly

In general, one can show

$$\frac{s_0}{s} = \sum_{\ell=0}^{\infty}\left(\frac{r}{s_0}\right)^{\ell} P_\ell(\cos\theta)$$

[Thomas observed this]

$$\frac{s_0}{s} - \frac{z}{s_0} = 1 - \cancel{\frac{r}{s_0}P_1(\cos\theta)} + \frac{r^2}{s_0^2}P_2(\cos\theta) - \cancel{\frac{r}{s_0}P_1(\cos\theta)} + \ldots$$

$$U_t \approx -\frac{GM_J m}{s_0}\left(1 + \frac{r^2}{s_0^2}P_2(\cos\theta) + \ldots\right)$$

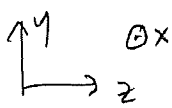
ignore irrelevant constant

Re-express in Cartesian coordinate

$$U_t = -\frac{G m_s m}{s_0^3}\, r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$= -\frac{G m_s m}{s_0^3}\left(\frac{3}{2}\cdot z^2 - \frac{1}{2}\left[x^2+y^2+z^2\right]\right)$$

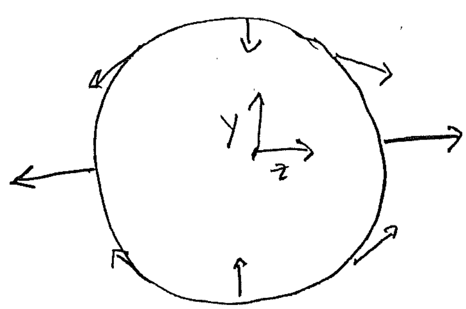
$$= -\frac{G m_s m}{s_0^3}\left(z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)$$



$$F_{tz} = -\frac{\partial U_t}{\partial z} = +\frac{2 G m_s m}{s_0^3}\, z$$

$$F_{ty} = -\frac{\partial U_t}{\partial x} = -\frac{G m_s m}{s_0^3}\, y$$

[Recall French figure]



Will cause oceans to form two bulges

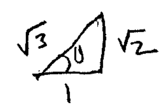
first explained by Newton

[Compare tidal force to earth gravity

$$\frac{F_t}{F_e} \sim \frac{\left(\frac{m_s r_e}{s_0^3}\right)}{\left(\frac{m_e}{r_e^2}\right)} = \frac{m_s}{m_e}\left(\frac{r_e}{s_0}\right)^3 \sim (2E5)(4E-5)^3 \sim 1.5E-8$$
]

[optional: $0 = \vec{r}\cdot\vec{F} = \cos\theta\,(2\cos\theta) + \sin\theta\,(-\sin\theta)$

$$= 3\cos^2\theta - 1 \quad\Rightarrow\quad \cos\theta = \frac{1}{\sqrt{3}} = 54.7^\circ$$



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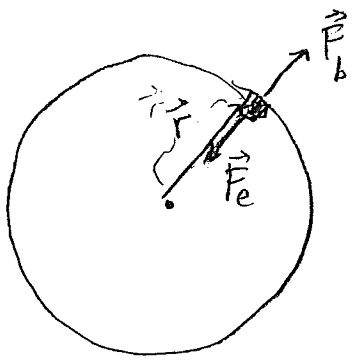
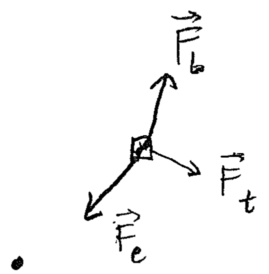
Consider force on parcel of water in equilibrium on earth's surface

$$\vec{F}_e = m\vec{g}_e \qquad \vec{g}_e = -\frac{GM_e}{r_e^2}\hat{r}$$



W/out tidal force, water is supported by radial buoyant force

$$\vec{F}_b = -\vec{F}_e$$



w/ tidal force, water still supported by buoyant force but no longer radial (necessarily)

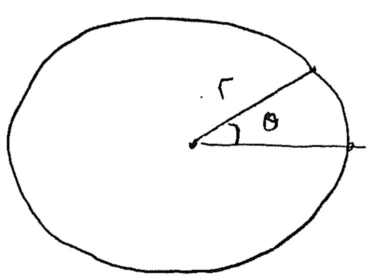
$$\vec{F}_b = -\vec{F}_e - \vec{F}_t$$

Since $\vec{F}_e = -\vec{\nabla}U_e$ where $U_e = mg_e h$, $h = r - r_e$

and $\vec{F}_t = -\vec{\nabla}U_t$

$$\Rightarrow \vec{F}_b = \vec{\nabla}U \quad \text{where} \quad U = U_e + U_t$$

Buoyant force must be normal to oceans surface (liquid can't support shear), but $\vec{\nabla}U$ is normal to equipotential surface, so surface of ocean is an equipotential surface of U



$$r = r_e + h(\theta)$$

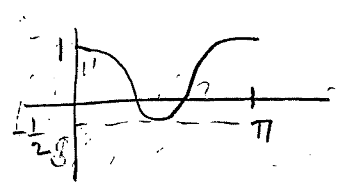
$$U = U_e + U_t$$

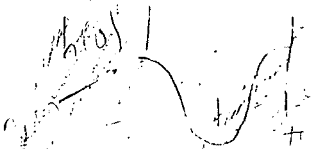
$$U = m g_e h(\theta) - \frac{G M_s m}{s_0^3} r^2 P_2(\cos\theta)$$

$$\approx m\left(\frac{G m_e}{r_e^2} h - \frac{G M_s}{s_0^3} r_e^2 P_2(\cos\theta)\right)$$

$$U = \text{const} \Rightarrow h(\theta) = \frac{M_s}{m_e}\frac{r_e^4}{s_0^3} P_2(\cos\theta) + \text{const}$$

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$





difference between $\theta = 0$ and $\theta = \frac{\pi}{2}$

$$\Delta P_2 = \frac{3}{2}$$

$$\Delta h = \frac{3}{2}\frac{m_s}{m_e}\left(\frac{r_e^4}{s_0^3}\right) = (3.8E{-}8)\, r_e$$

$$\simeq 25 \text{ cm}$$

$m_e = 5.97E24\,kg$, $m_s = 1.99E30\,kg$ $\Big\}$ $\frac{m_s}{m_e} = 3.3E5$

$r_e = 6.38E6\,m$, $s_0 = 1.50E11\,m$ $\Big\}$ $\left(\frac{r_e}{s_0}\right)^3 = 7.7E{-}14$

$$\left(\frac{r_e}{s_0}\right) = 4.25E{-}5$$

moon also causes tides

$$\Delta h = \frac{3}{2}\,\frac{m_m}{m_e}\left(\frac{r_e^4}{d_m^3}\right) = (8.5E-8)\, r_e \approx 54 \text{ cm}$$

$m_m = 7.35E22$ kg

$d_m = 3.84E8$ m

$\frac{m_m}{m_e} = 0.0123$

$\left(\frac{r_e}{d_m}\right)^3 = 4.59E-6$

$\left(\frac{r_e}{d_m}\right) = 0.0166$

moon mass less, but much closer

when moon is new or full, tides reinforce $\Delta h = 79$ cm "spring tide"

New E m S

Full m E S

when moon is half, effects partially cancel $\Delta h = 29$ cm "neap tide"

m

E S

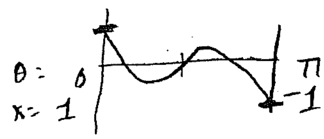
oral problem on tides

$$\frac{s_0}{s} = 1 + \frac{r}{s_0} P_1(\cos\theta) + \frac{r^2}{s_0^2} P_2(\cos\theta) + \frac{r^3}{s_0^3} P_3(\cos\theta) + \ldots$$

$$\Phi = \frac{G m_e}{r_e^2} h(\theta) - \frac{G M_s}{s_0}\left(\frac{r^2}{s_0^2} P_2(\cos\theta) + \frac{r^3}{s_0^3} P_3(\cos\theta)\right)$$

$$h(\theta) = \frac{m_s}{m_e}\left(\frac{r_e^4}{s_0^3} P_2(\cos\theta) + \frac{r_e^5}{s_0^4} P_3(\cos\theta)\right)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$



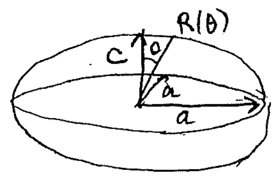
$$\Delta P_3 = 1 - (-1) = 2$$

$$\Delta h = \frac{2 m_s}{m_e}\left(\frac{r_e^5}{s_0^4}\right) = 1.4 \times 10^{-5}\,m = (2.2 \times 10^{-12}) R_e$$

$$\Delta h = \frac{2 m_n}{m_e}\left(\frac{r_e^5}{s_0^4}\right) = 1.2\,cm = (1.9 \times 10^{-9}) R_e$$

(1-24-24)

Consider an ellipsoid of revolution



$R(0) = c$ $\quad a > c$ oblate

$R(\frac{\pi}{2}) = a$ $\quad a < c$ prolate

$$1 = \frac{z^2}{c^2} + \frac{x^2+y^2}{a^2} = R^2\left(\frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta}{a^2}\right) = R^2\,\frac{a^2\cos^2\theta + c^2\sin^2\theta}{a^2c^2}$$

$$R(\theta) = \frac{ac}{\sqrt{a^2\cos^2\theta + c^2\sin^2\theta}}$$

Consider a sphere of equal volume $\quad R_0^3 = a^2 c$

Define $\quad \epsilon = \dfrac{a-c}{R_0}$

Trefil, 27: $\quad a = 6378$ km, $c = 6357$ km $\quad \boxed{a - c = 21 \text{ km}}$

$R_0 = 6371$ km

$\epsilon = 3.30 \times 10^{-3}$

(Fetter & Walecka (prob 2.7) state $\epsilon = 3.35 \times 10^{-3}$)

$$R_0^3 = a^2 c$$

$$= (R_0 + \delta a)^2 (R_0 + \delta c) \Rightarrow \delta c = -2\,\delta a$$

$$\epsilon = \frac{a-c}{R_0} = \frac{\delta a - \delta c}{R_0} = \frac{3\,\delta a}{R_0} \Rightarrow \delta a = \frac{R_0}{3}\epsilon$$

$$\delta c = -\frac{2R_0}{3}\epsilon$$

$$a = R_0\left(1 + \tfrac{1}{3}\epsilon + O(\epsilon^2)\right)$$

$$c = R_0\left(1 - \tfrac{2}{3}\epsilon + O(\epsilon^2)\right)$$

$$R(\theta) = \frac{ac}{\sqrt{a^2\cos^2\theta + c^2(1-\cos^2\theta)}}$$

$$= \frac{R_0^2\left(1+\frac{\epsilon}{3}\right)\left(1-\frac{2}{3}\epsilon\right)}{\sqrt{R_0^2\left(1+\frac{2\epsilon}{3}\right)\cos^2\theta + R_0^2\left(1-\frac{4\epsilon}{3}\right) - R_0^2\left(1-\frac{4\epsilon}{3}\right)\cos^2\theta}}$$

$$= \frac{R_0\left(1-\frac{\epsilon}{3}\right)}{\sqrt{1 - \frac{4\epsilon}{3} + 2\epsilon\cos^2\theta}}$$

$$= R_0\left(1 - \epsilon\cos^2\theta + \frac{\epsilon}{3}\right)$$

$$= R_0\left(1 - \frac{2}{3}\epsilon\left[\frac{3}{2}\cos^2\theta - \frac{1}{2}\right]\right)$$

$$\boxed{R(\theta) = R_0\left(1 - \frac{2\epsilon}{3}P_2(\cos\theta)\right)}$$

Assume $\epsilon << R_0$

Treat earth as sphere of mass M and radius R_0

plus a shell of radius R_0 and surface density $\sigma(\theta)$

$$\sigma(\theta) = \rho_s\left[R(\theta) - R_0\right] = -\frac{2\epsilon}{3}\rho_s R_0 P_2(\cos\theta)$$

(Note that the total mass of the shell is zero.)

ρ_s = density of earth near surface

The gravitational potential of the sphere of mass M has different form inside or outside, but at points near the surface, it is approximately given by

$$\Phi_{gh} = \left(\frac{GM}{R_0^2}\right)(r - R_0) \qquad \text{ie } gh$$

eg can write $\Phi = -\frac{GM}{r} = \frac{-GM}{R_0 + (r - R_0)} = \frac{-GM}{R_0}\left(1 - \frac{r-R_0}{R_0} + \ldots\right)$

$$= \frac{GM}{R_0^2}(r - R_0) + (\text{const}) \leftarrow \text{ignore}$$

Thus on the surface of the ellipsoid,

$$\Phi_{sph} = \left(\frac{GM}{R_0^2}\right)(R(\theta) - R_0) = -\frac{2\epsilon}{3}\frac{GM}{R_0}P_2(\cos\theta)$$

The gravitational potential of a shell at $|\vec{r}'| = R_0$ is given by

$$\Phi_{shell}(\vec{r}) = -G \int \frac{\sigma R_0^2 \, d\Omega'}{|\vec{r}-\vec{r}'|}$$

For $|\vec{r}| > |\vec{r}'|$ we have

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_\ell \left(\frac{R_0}{r}\right)^\ell P_\ell(\cos\theta) P_\ell(\cos\theta')$$

while for $|\vec{r}| < |\vec{r}'|$ we have

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{R_0} \sum \left(\frac{r}{R_0}\right)^\ell P_\ell(\cos\theta) P_\ell(\cos\theta')$$

For $|\vec{r}| \approx R_0$, these both reduce to

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{R_0} \sum P_\ell(\cos\theta) P_\ell(\cos\theta')$$

Since $\sigma = -\frac{2\epsilon}{3} \rho_s R_0 \, P_2(\cos\theta)$ we have

$$\int P_2(\cos\theta') \sum_\ell P_\ell(\cos\theta) P_\ell(\cos\theta') \, d\Omega'$$

$$= \sum_\ell P_\ell(\cos\theta) \, 2\pi \underbrace{\int P_2(\cos\theta') P_\ell(\cos\theta') \, d(\cos\theta')}_{\frac{2}{2\ell+1}\delta_{\ell 2}}$$

$$= \frac{4\pi}{5} P_2(\cos\theta)$$

Hence $\Phi_{shell}(R_0, \theta) = -GR_0 \left(-\frac{2\epsilon}{3}\rho_s R_0\right)\left(\frac{4\pi}{5}\right) P_2(\cos\theta)$

$$= \frac{8\pi}{15} G R_0^2 \rho_s \epsilon \, P_2(\cos\theta)$$

$$= \frac{GM}{R_0} \underbrace{\left(\frac{8\pi}{15} \frac{R_0^3 \rho_s}{M} \epsilon\right)}_{J_2 \text{ in Fetter Walecka prob 27}} P_2(\cos\theta)$$

$J_2 = \frac{2}{5} \frac{\rho_s}{\rho_e} \epsilon = 1.083 \times 10^{-3}$

ρ_e = avg earth density

Thus gravitational potential on the ellipsoidal surface is

$$\Phi = \Phi_{sph} + \Phi_{sh}$$

$$= \frac{GM}{R_0}\left[-\frac{2\epsilon}{3} + \underbrace{\frac{2}{5}\frac{\rho_s}{\rho_e}\epsilon}_{J_2 = 1.083\times10^{-3}}\right] P_2(\cos\theta)$$

Fettr Walecka prob 2.7

$$\boxed{\Phi = \frac{GM}{R_0}\left(-\frac{2\epsilon}{3}\right)\left(1-\frac{3\rho_s}{5\rho_e}\right)P_2(\cos\theta)}$$

If $\rho_s = \rho_e$ then $\left(1-\frac{3}{5}\right) = 0.4$

If $\rho_s = \frac{1}{5}\rho$ then $\left(1-\frac{3}{25}\right) \approx 0.9$

↑ water

If $\rho_s = \frac{2}{5}\rho$ then $\left(1-\frac{6}{25}\right) \approx 0.8$

↑ crust?

$$\Phi = \frac{4}{15}\frac{GM}{R_0}\left(\frac{a-c}{R_0}\right)P_2(\cos\theta)$$

cf Richard Fitzpatrick eq (911)

<u>Rotating earth</u>

$$|\vec{F}_{centrif}| = m\omega^2 r_\perp$$

$$\Phi_{centrif} = -\frac{1}{2}\omega^2 r_\perp^2$$

$$= -\frac{1}{2}\omega^2 R_0^2 \sin^2\theta$$

$$= (\text{const}) + \frac{1}{3}\omega^2 R_0^2 P_2(\cos\theta$$

$$\Phi_{centrif} + \Phi_{ellip} = \text{const}$$

$$\Rightarrow \quad \frac{2\epsilon}{3}\left(\frac{GM}{R_0}\right)\left(1 - \frac{3}{5}\frac{\rho_s}{\rho_e}\right) = \frac{1}{3}\omega^2 R_0^2$$

$$\underbrace{\epsilon}_{\substack{\text{observed} \\ 3.3\times10^{-3}}} = \underbrace{\frac{1}{2}\frac{\omega^2 R_0^3}{GM}}_{1.7\times10^{-3}}\left(1 - \frac{3}{5}\frac{\rho_s}{\rho_e}\right)^{-1}$$

Fetter Walecka, prob 2.7

$$\Phi = \Phi_{sph} + \Phi_{ellip} + \Phi_{cent}$$

$$= \frac{GM}{R_0}\left(-\frac{2\epsilon}{3} + J_2 + \frac{\omega^2 R_0^3}{3GM}\right) P_2(\cos\theta)$$

$$\underbrace{\epsilon}_{3.3\times10^{-3}} = \underbrace{\frac{3}{2}J_2}_{\substack{1.6\times10^{-3} \\ \text{F+W}}} + \underbrace{\frac{\omega^2 R_0^3}{2GM}}_{1.7\times10^{-3}}$$

But $J_2 = \frac{2}{5}\frac{\rho_s}{\rho_e}\epsilon$

$$\frac{\rho_s}{\rho_e} = \frac{5}{2}\frac{J_2}{\epsilon} = \underline{\underline{0.8}}$$

Tidal deformat[ion]

$$\Phi_{tid} = -\frac{G M_s R_0^2}{s_0^3} P_2(\cos\theta)$$

$$\Phi = \Phi_{sph} + \Phi_{ellip} + \Phi_{tid}$$

$$= \left[\frac{G M_e}{R_0}\left(-\frac{2\epsilon}{3}\right)\left(1-\frac{3\rho_s}{5\rho_e}\right) - \frac{G M_s R_0^2}{s_0^3}\right] P_2(\cos\theta)$$

$$\epsilon = \underbrace{-\frac{3}{2}\frac{M_s}{M_e}\left(\frac{R_0}{s_0}\right)^3}\left(1-\frac{3\rho_s}{5\rho_e}\right)^{-1}$$

~1 m for moon + sun