3120 - D
2024

In solving one-body problems in which external forces act on a mass

We are ignoring the elephant in the room.

All forces result from the interaction between two (or more) objects in which Both objects are affected.

In the cases above one of the ~~bodies~~ objects happens to be much more massive than the other (eg earth, proton)

We will now consider the more general 2-body problem involving just 2 objects, & then we'll see why often we can neglect one if it is much more massive

All forces result from interactions <u>between</u> objects

standard convention:

$\vec{F}_{12}$ = force on 1 by 2

$\vec{F}_{21}$ = force on 2 by 1

Newton's 3rd law (action reaction)

$$\vec{F}_{21} = -\vec{F}_{12}$$

First, isolated system of 2 interacting particles

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{12}$$

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{21}$$

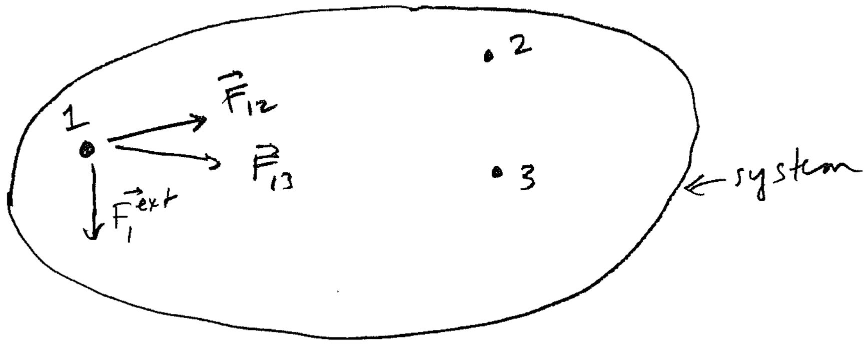
$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \vec{F}_{12} + \vec{F}_{21} = 0$$

$$\vec{p}_1 + \vec{p}_2 = \text{const}$$

[individual momenta change but total momentum conserved]

System of particles

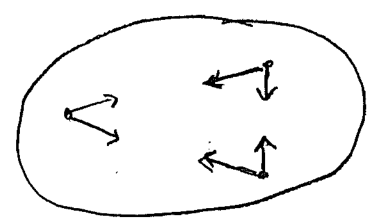
Consider N objects, interacting w/ one another and possibly also w/ objects outside the system



$$\frac{d\vec{p}_i}{dt} = \vec{F}_i \qquad \text{(net force on } i\text{)}$$

$$\vec{F}_i = \vec{F}_i^{ext} + \sum_{\substack{j \\ (j \neq i)}} \vec{F}_{ij}$$

Define system (or total) momentum



$$\vec{p}^{\,sys} = \sum_{i=1}^{N} \vec{p}_i$$

$$\frac{d\vec{p}^{\,sys}}{dt} = \sum \frac{d\vec{p}_i}{dt} = \sum \vec{F}_i$$

$$= \sum_i \vec{F}_i^{ext} + \sum_i \sum_{j \neq i} \vec{F}_{ij}$$

relabel $i \leftrightarrow j$

$$= \vec{F}^{ext} + \underbrace{\sum_i \sum_{j<i} \vec{F}_{ij} + \sum_i \sum_{j>i} \vec{F}_{ij}}_{\sum_{\text{pairs } j<i} (\vec{F}_{ij} + \vec{F}_{ji})}$$

But $\vec{F}_{ij} + \vec{F}_{ji} = 0$ by 3rd law

$$\boxed{\frac{d\vec{p}^{\,sys}}{dt} = \vec{F}^{ext}} \quad \text{where } \vec{F}^{ext} = \sum \vec{F}_i^{ext}$$

[internal interactions cancel]

A system is isolated if $\vec{F}^{ext} = 0$

$$\frac{d\vec{p}^{sys}}{dt} = 0 \quad \Rightarrow \quad \vec{p}^{sys} = \text{const}$$

Total momentum of an isolated system is conserved

3 forms of isolation

1) truly (or perhaps approximately)

2) functionally (if external forces cancel, eg. gravity + normal)

3) momentarily (collision)

$$d\vec{p}^{sys} = \vec{F}^{ext}\, dt \underset{\text{if } dt \to 0}{\longrightarrow} 0$$

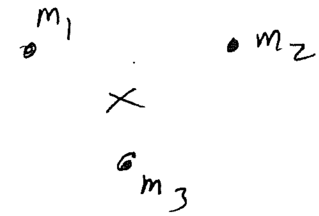
Mom. conservation can hold for some components & not others

$$\frac{dp_z^{sys}}{dt} = F_z^{ext} \neq 0 \quad \text{(gravity)}$$

$$\frac{dp_x^{sys}}{dt} = F_x^{ext} = 0$$

[A very useful concept is]

Center of mass



total mass $M = \sum m_i = \int dm$

center of mass $\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i = \frac{1}{M} \int dm\, \vec{r}$

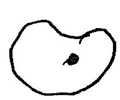
"mass-weighted average position"

$$\vec{p}^{\,sys} = \sum \vec{p}_i = \sum m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt}\left(\sum m_i \vec{r}_i\right) = \frac{d}{dt}(M \vec{r}_{cm}) = M \vec{v}_{cm}$$

$$\frac{d\vec{p}^{\,sys}}{dt} = M \vec{a}_{cm} = \vec{F}^{ext}$$

center of mass moves as a particle of mass M
acted on by $\vec{F}^{ext}$

[This is why we treat extended objects, like bricks or planets as point-like particles, ignoring internal forces]

Demo: 

Firework?

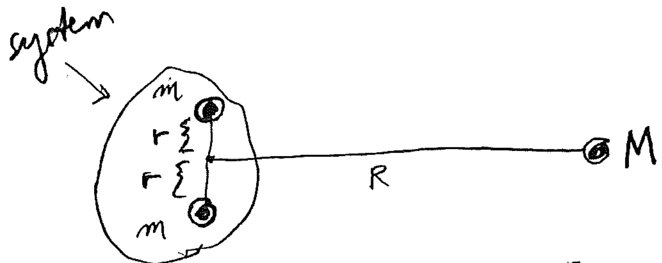
Important subtlety: $\vec{F}^{ext} = \sum \vec{F}_i^{ext}(\vec{r}_i)$

forces evaluated at positions of individual particle

not at $\underline{cm}$

(doesn't matter if external force is uniform, e.g. $\vec{F} = m\vec{g}$

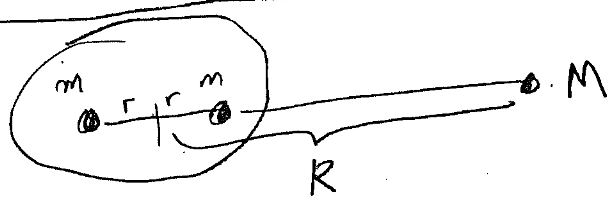
but results in tidal force on larger scale)



$$F_x^{ext} = 2\frac{GMm}{R^2+r^2}\left(\frac{R}{\sqrt{R^2+r^2}}\right)$$

$$= \frac{2GMm}{R^2}\left(1+\frac{r^2}{R^2}\right)^{-3/2}$$

$$= \frac{GM(2m)}{R^2}\left(1-\frac{3}{2}\frac{r^2}{R^2}+\dots\right)$$ slightly less than $\frac{GM(2m)}{R^2}$



$$F_x^{ext} = \frac{GMm}{(R+r)^2} + \frac{GMm}{(R-r)^2}$$

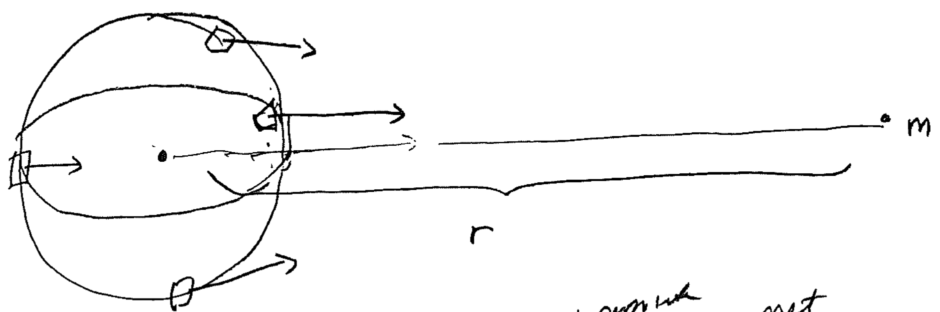
$$= \frac{GMm \cdot 2(R^2+r^2)}{(R^2-r^2)^2} =$$

$$= \frac{GM(2m)}{R^2}\left(1+\frac{3r^2}{R^2}+\dots\right)$$ slightly greater

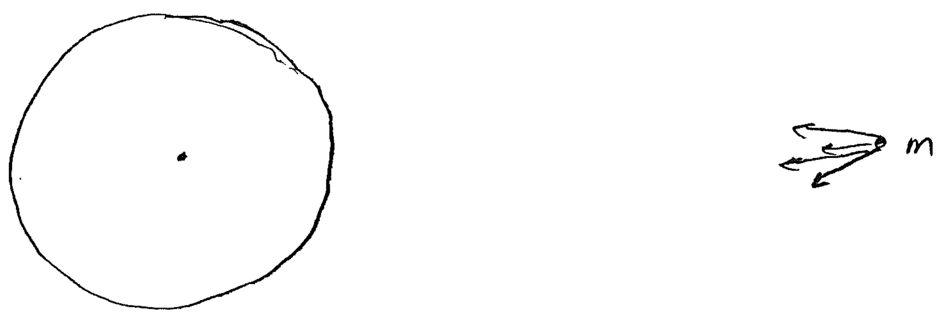
$$F_x^{ext} = \frac{GM(4m)}{R^2}\left(1+O\left(\frac{r^4}{R^4}\right)\right)$$

Gravitational force on a sphere

Consider the force exerted on a spherical shell of uniformly distributed mass M by a point mass m outside the shell and located a distance r from the center of the shell



By 3rd law, this force is equal + opposite to the net force exerted by the shell on the point mass

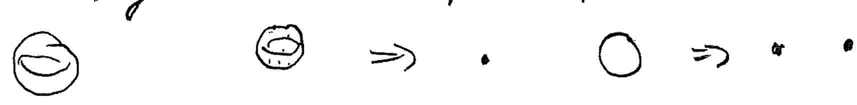


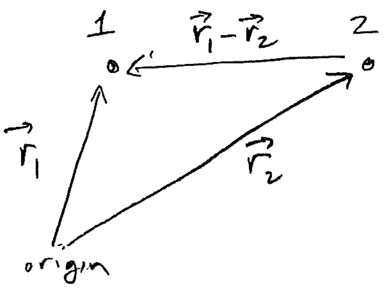
In Phys 3000, we calculated this force to be exactly $|\vec{F}| = \frac{GMm}{r^2}$

Therefore $\left|\underset{\substack{\uparrow \\ \text{on shell}}}{\vec{F}^{ext}}\right| = \frac{GMm}{r^2}$

Since $\vec{F}^{ext} = M\ddot{\vec{r}}_{cm}$, the cm acts <u>exactly</u> as if the shell were a point mass at $\vec{r}_{cm}$.

(Now replace point mass by a shell, or sphere of mass)





$\vec{F}_{12}$ = force on 1 by 2

depends not on $\vec{r}_1$ + $\vec{r}_2$ separately, but only on

$\vec{r}_1 - \vec{r}_2 = \vec{r}_{12}$ = position of 1 relative to 2

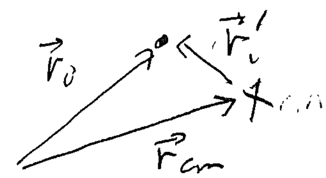
write $\vec{F}_{12}(\vec{r}_{12})$

[makes sense because]
Force cannot depend on arbitrary choice of origin

Translation invariance of laws of physics

[Not necessarily true that $\vec{F}_{12} \parallel \vec{r}_{12}$
unless $\vec{F}$ is central force
ie $\vec{F} = -\vec{\nabla} U$ and $U = U(|\vec{r}_{12}|)$]

Center of mass frame
(not necessarily IRF)



Define positions relative to cm

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i'$$

that is

$$\vec{r}_i' = \vec{r}_i - \vec{r}_{cm}$$ — observe $\vec{F}_{12}(\vec{r}_1 - \vec{r}_2) = \vec{F}_{12}(\vec{r}_1' - \vec{r}_2')$

$$\vec{v}_i' = \vec{v}_i - \vec{v}_{cm}$$

$$a_i' = \vec{a}_i - \vec{a}_{cm}$$

Some obvious things:

position of cm in cm frame is zero.

$$\vec{r}_{cm}' = \frac{1}{M}\sum m_i \vec{r}_i' = \frac{1}{M}\sum m_i(\vec{r}_i - \vec{r}_{cm}) = \vec{r}_{cm} - \frac{\sum m_i}{M}\vec{r}_{cm} = 0$$

velocity of cm in cm frame is zero

$$\vec{v}_{cm}' = 0$$

total moment in cm frame is zero

$$\vec{p}'^{\,sys} = \sum m_i \vec{v}_i' = \sum m_i(\vec{v}_i - \vec{v}_{cm}) = \underbrace{\sum m_i \vec{v}_i}_{\vec{p}^{\,sys}} - \underbrace{\left(\sum m_i\right)}_{M}\vec{v}_{cm} = 0$$

cm frame = zero momentum frame

[only useful if also IRF]

Recall

$$\frac{d\vec{p}_i}{dt} = m_i \vec{a}_i = \vec{F}_i^{ext}(\vec{r}_i) + \sum_{j \neq i} \vec{F}_{ij}(\vec{r}_i - \vec{r}_j) \quad \text{[in IRF]}$$

Express this in cm frame

$$m_i(\vec{a}_i' + \vec{a}_{cm}) = \vec{F}_i^{ext}(\vec{r}_i' + \vec{r}_{cm}) + \sum_{j \neq i} \vec{F}_i(\vec{r}_i' - \vec{r}_j')$$

That is

$$\boxed{m_i \vec{a}_i' = -m_i \vec{a}_{cm} + \vec{F}_i^{ext}(\vec{r}_i' + \vec{r}_{cm}) + \sum_{j \neq i} \vec{F}_i(\vec{r}_i' - \vec{r}_j')}$$

acceleration of i in cm frame

acceleration of the CM (relative to some IRF)

can regard this as a pseudoforce resulting from being in an accel. ref frame

examine 3 cases

① isolated system

② uniform external force

③ non uniform external force

① Isolated system : $\vec{F}^{ext} = 0$

$$m_i \vec{a}_i = \sum_{j \neq i} F_{ij}(\vec{r}_i - \vec{r}_j) \quad (*) \quad \text{in IRF}$$

$$M \vec{a}_{cm} = \vec{F}^{ext} = 0 \Rightarrow \vec{a}_{cm} = 0$$

$$\vec{V}_{cm} = \vec{V}_0 \quad \text{(const)}$$

$$\vec{r}_{cm} = \vec{v}_0 t + \vec{r}_0$$

cm moves w const velocity

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cm frame: $\vec{r}_i{}' = \vec{r} - \vec{r}_{cm} = \vec{r} - \vec{v}_0 t - \vec{r}_0$

This is a boost into a const velocity IRF

$$m_i \vec{a}_i{}' = \sum \vec{F}_{ij}(\vec{r}_i{}' - \vec{r}_j{}') \quad (**) \quad \text{in cm frame}$$

NB. (*) and (**) have same form
Newton's 2nd law invariant under a boost
(principle of relativity)

② external force indep of position

(eg uniform grav. field)

$$\vec{F}_i^{ext}(\vec{r}_i) = m_i \vec{g} \qquad \forall\ \vec{g} = \text{const}$$

$\downarrow \vec{g}$

$$m_i \vec{a}_i = m_i \vec{g} + \sum_{j \neq i} \vec{F}_{ij}(\vec{r}_i - \vec{r}_j) \qquad \text{[IRF]} \qquad (*)$$

In cm frame

$$m_i \vec{a}_i' = -m_i \vec{a}_{cm} + m_i \vec{g} + \sum_{j \neq i} \vec{F}_{ij}(\vec{r}_i' - \vec{r}_j')$$

Now
$$M \vec{a}_{cm} = \vec{F}^{ext} = \sum m_i \vec{g} = M \vec{g}$$

$$\Rightarrow \vec{a}_{cm} = \vec{g} \qquad \text{(cm frame is accelerating downward) "freely falling"}$$

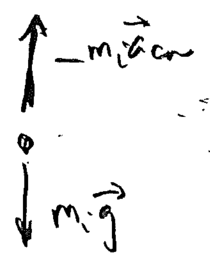
Thus
$$m_i \vec{a}_i' = \sum \vec{F}_{ij}(\vec{r}_i' - \vec{r}_j') \qquad (**)$$

eqns in the (freely falling) cm frame are same as in an isolated system (i.e. external grav field goes away)

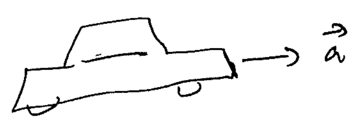
NB eom $(*)$ + $(**)$ not the same because cm frame is an accelerated frame not IRF

Another way to view this

$$m_i \vec{a}_i{}' = \underbrace{-m_i \vec{a}_{cm}}_{\text{Treat this as pseudo-force due to accelerating ref. frame (not IRF)}} + \underbrace{m_i \vec{g}}_{\text{external force}} + \sum_{j \neq i} \vec{F}_{ij}(\vec{r}_i{}' - \vec{r}_j{}')$$

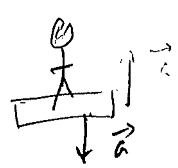


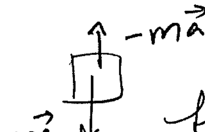
[Think of acceleration in a car



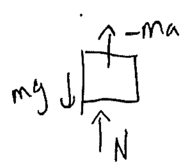
you are "pushed" back into cushions.

or going down in an elevator:



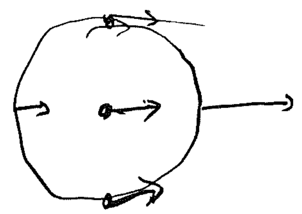


feel lighter]



$N = mg - ma$

Case (3) non uniform external force
(e.g. non uniform grav. field $\vec{g}(\vec{r})$)



$\circ M$

$$m_i \vec{a}_i' = \underbrace{-m_i \vec{a}_{cm}}_{\text{pseudo force}} + \underbrace{m_i \vec{g}(\vec{r}_i)}_{\text{external force}} + \sum_{j \neq i} F_{ij}(\vec{r}_i - \vec{r}_j)$$

where $\vec{a}_{cm} = \frac{1}{M} \vec{F}^{ext} = \frac{1}{M} \sum m_i \vec{g}(\vec{r}_i)$

Recall for a spherical mass $\sum m_i \vec{g}(\vec{r}_i) = M \vec{g}(\vec{r}_{cm})$

so $\vec{a}_{cm} = \vec{g}(\vec{r}_{cm})$

$$m_i \vec{a}_i' = \underbrace{m_i \left(\vec{g}(\vec{r}_i) - \vec{g}(\vec{r}_{cm}) \right)}_{\text{residual "tidal" force}} + \sum F_{ij}(\vec{r}_i - \vec{r}_j)$$

depends on differential gravitational fields across the object

i.e. difference between grav field + average grav field

(11-23-23)

Two body problem [now done as an oral problem] [will return to this later in context of orbit eqn]

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1^{\,ext} + \vec{F}_{12}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \vec{F}_2^{\,ext} + \vec{F}_{21} \leftarrow = -\vec{F}_{12}$$

Then

$$\frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2) = \vec{F}_1^{\,ext} + \vec{F}_2^{\,ext} = \vec{F}^{ext}$$

$$\frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = \left(\frac{\vec{F}_1^{\,ext}}{m_1} - \frac{\vec{F}_2^{\,ext}}{m_2}\right) + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}$$

Define $\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$

$$\begin{cases} \vec{r}_1 = \vec{r}_{cm} + \frac{m_2}{M}(\vec{r}_1 - \vec{r}_2) \\ \vec{r}_2 = \vec{r}_{cm} + \frac{m_1}{M}(\vec{r}_2 - \vec{r}_1) \end{cases}$$

Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$$M\frac{d^2\vec{r}_{cm}}{dt^2} = \vec{F}^{ext}$$

$$\mu \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2) + \left(\frac{m_2}{M}\vec{F}_1^{\,ext} - \frac{m_1}{M}\vec{F}_2^{\,ext}\right)$$

If $\vec{F}_i^{\,ext} = m_i\vec{g}$, then eqns decouple

$$\frac{d^2\vec{r}_{cm}}{dt^2} = \vec{g}$$

$$\mu\frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2)$$

If uniform external electric field, eqns still decouple (Sommerfeld Atomic Structure, 276)
but $\vec{r}_1 - \vec{r}_2$ depends on external field as well as F_{12}
$\longrightarrow (q_1 m_2 - q_2 m_1)\frac{E}{M}$

(Soln to oral problem)

Kinetic energy of 2-body system

$$T^{sys} = T_1 + T_2$$

$$= \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2}$$

$\vec{r}_1'$

Using $\begin{cases} \vec{v}_1 = \vec{v}_{cm} + \dfrac{m_2}{M}\vec{v}_{12} \\ \vec{v}_2 = \vec{v}_{cm} - \dfrac{m_1}{M}\vec{v}_{12} \end{cases}$ $\quad \dot{\vec{r}}_2'$ $\qquad \vec{v}_{12} = \vec{v}_1 - \vec{v}_2$

one has

$$T^{sys} = \frac{1}{2} m_1 \left(\vec{v}_{cm} + \frac{m_2}{M}\vec{v}_{12}\right)^2 + \frac{1}{2} m_2 \left(\vec{v}_{cm} - \frac{m_1}{M}\vec{v}_{12}\right)^2$$

$$= \frac{1}{2}(m_1+m_2)\vec{v}_{cm}^{\,2} + 0 + \frac{1}{2}\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right)\vec{v}_{12}^{\,2}$$

$$= \frac{1}{2} M \vec{v}_{cm}^{\,2} + \underbrace{\frac{1}{2}\mu \vec{v}_{12}^{\,2}}$$

Kinetic energy of equivalent 1-body problem

Also, kinetic energy of system in the cm frame

[Note: $\frac{m_2}{M}\dot{\vec{r}}$ & $-\frac{m_1}{M}\dot{\vec{r}}$ are velocities $\dot{\vec{r}}_1'$ and $\dot{\vec{r}}_2'$ in cm frame]

2018 notes

omitted in 2024

Sommerfeld Mechanics

Mechanical energy of isolated 2-body system

$$E_{mech} = \frac{1}{2} M \vec{V}_{cm}^{\,2} + \left[\frac{1}{2} \mu \vec{v}_{12}^{\,2} + U_{12}(\vec{r}_{12}) \right]$$

Consider an elastic collision. Well before and well after collision

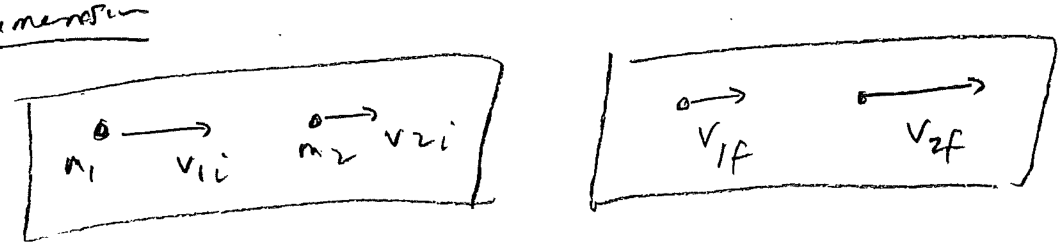
$U_{12} \to 0$, so $\quad E_{mech} = \frac{1}{2} M \vec{V}_{cm}^{\,2} + \frac{1}{2}\mu \vec{v}_{12}^{\,2}$

- $\vec{V}_{cm}$ is unchanged in collision [due to internal forces]
- Since E_{mech} conserved in elastic collision, $\vec{v}_{12}^{\,2}$ is unchanged

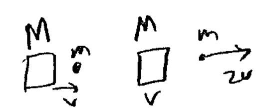
$$\Rightarrow \quad |\vec{v}_{rel,final}| = |\vec{v}_{rel,init}|$$

speed of recession = speed of approach

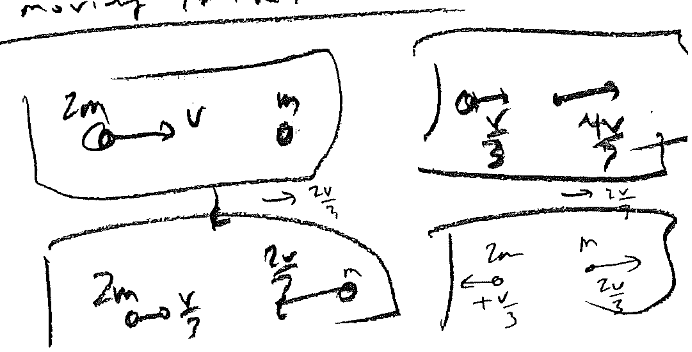
one dimension



- equal masses
- very unequal: ping pong ball + stationary truck
  stationary ping pong ball + moving truck.



Then Do eg



2018 notes

omit in 2024

inelastic collisions: internal forces are non conservative
so internal kinetic energy is lost to heat

$$\Rightarrow |\vec{v}_{rel,final}| < |\vec{v}_{rel,init}|$$

completely ~~extremely~~ inelastic: $\Rightarrow$ $|\vec{v}_{rel,final}| = 0$
objects stick together