PHYS 3120 Advanced mechanics Spring 2024

Problem sets 40%
Exams 60%

3 approaches to mechanics: Newtonian, Lagrangian, Hamiltonian

Newtonian mechanics of a point particle of mass m

2nd law $\frac{d\vec{p}}{dt} = \vec{F}^{net}$

$$\vec{F}^{net} = \sum_A \vec{F}^{(A)}$$

[vector sum of all forces A acting on m; gravity, electromagnetic, constraint, friction ...]

[call it net because can partially cancel out due to vector nature]

2nd law valid in any inertial reference frame

[what is IRF? one in which 2nd law holds! circular?]

1st law: An inertial reference frame is one in which, if $\vec{F}^{net} = 0$, then $\vec{p}$ = constant, ie $\vec{v}$ = const

[seems like special case of 2nd but it defines an IRF. Claim: IRF exists.]

A frame related to an IRF by a boost is also an IRF

[Therefore ∞ no. of IRFs exist]

[derivative defined by a limiting process but physicists like to treat it as ratio]

[can recast 2nd law as]

$$\underbrace{d\vec{p}}_{\text{infinitesimal change}} = \vec{F}\,dt$$

[$d\vec{p}$ = small (infinitesimal) change in momentum when net force acts for a small time]

$$= \sum \underbrace{\vec{F}^{(A)}dt}_{\text{"impulse", kick}}$$

$$\underbrace{\Delta\vec{p}}_{\text{finite change}} = \vec{p}_f - \vec{p}_i = \int d\vec{p} = \int_{t_i}^{t_f} \vec{F}\,dt$$

[impulse is infinitesimal for finite forces but can be finite in a collision
eg ball bounces off floor, or kick a ball
$dt \to 0$ but $F_N \to \infty$ so dp = finite]

[To self: model rigid surface as a stiff spring

$$|\vec{F}| = k|x| = kA\cdot\sin\omega t$$

$$\int_0^{T/2} F\,dt = \frac{kA}{\omega}\cdot 2 = 2m\omega A = 2mv$$
]

Nonrelativistic momentum

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} = m\dot{\vec{r}}$$

Then change in momentum

$$\frac{d\vec{p}}{dt} = m\frac{d^2\vec{r}}{dt^2} = m\ddot{\vec{r}} = \vec{F}$$

[2nd most famous eqn in physics]

$$\frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m}$$

solution requires knowledge of $\vec{F}$ and two initial conditions $\vec{r}_0$ and $\vec{v}_0$

Example: constant (time) uniform (space) gravitational field $\vec{F} = m\vec{g}$

$$\vec{a} = \vec{g}$$

$$\vec{v} = \int \vec{a}\, dt = \vec{g}t + \vec{v}_0$$

$$\vec{r} = \int \vec{v}\, dt = \frac{1}{2}\vec{g}t^2 + \vec{v}_0 t + \vec{r}_0$$

projectile motion

[But usually $\vec{F}$ not specified as fcn of t but depends on $\vec{r}$ and/or $\vec{v}$, so can't just integrate]

[To self: relativistic particle in const electric field

$$\frac{d}{dt}(m\gamma v) = qE \Rightarrow \frac{v^2}{1-v^2} = \left(\frac{qEt}{m}\right)^2 \Rightarrow \frac{v}{c} = \frac{\frac{qEt}{m}}{\sqrt{1+\left(\frac{qEt}{m}\right)^2}}$$

]

[besides momentum, another big concept is energy]

Work-energy theorem

Nonrelativistic kinetic energy

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

Then change in kinetic energy

$$\frac{dT}{dt} = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

Infinitesimal change in kinetic energy

$$dT = \underbrace{\vec{F} \cdot d\vec{r}}_{\text{net work done on point mass}} = \sum_A \underbrace{\vec{F}^{(A)} \cdot d\vec{r}}_{\text{work done by each force acting}} \quad \text{[scalar sum]}$$

$$d\vec{p} = \vec{F}\, dt$$ [work : kinetic energy :: impulse : momentum]

[Let's discuss work done by different type of forces]

Classification of forces

1. conservative
2. constraint
3. all others (eg. dissipative)

① Conservative forces

(e.g., electrostatic, gravity)

Equivalent statements [PHYS 3000]

(a) work done $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}$ independent of path from $\vec{r}_1$ to $\vec{r}_2$

(b) $\oint \vec{F}\cdot d\vec{r} = 0$ no work done around closed loop

(c) $\vec{\nabla}\times\vec{F} = 0$

(d) $\vec{F} = -\vec{\nabla}U$ where U = potential energy function

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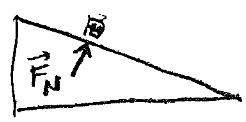
Change in U from $\vec{r}$ to $\vec{r}+d\vec{r}$.

$$dU = U(\vec{r}+d\vec{r}) - U(\vec{r})$$
$$= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$
$$= \vec{\nabla}U\cdot d\vec{r}$$
$$= -\vec{F}\cdot d\vec{r}$$

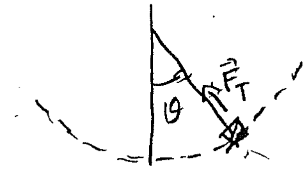
Work done by conservative force $\vec{F}\cdot d\vec{r} = -dU$

② Forces of constraint

– constrain particle to move along a specified path or surface



normal force



tension

Constraint forces are reactive

Their magnitudes are, a priori unknown

[depends on other forces & on the motion]

They are what they need to be to constrain the motion.

Must have $F_N \geq 0$ and $F_T \geq 0$

otherwise motion is no longer constrained.

[can't push a string]

Usually, constraint forces act perpendicular to displacement

In this case, they do no work: $\vec{F} \cdot d\vec{r} = 0$

(Caveat: Moving constraints can do work)

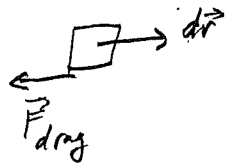
[Exceptions: moving constraints
eg rising elevator floor
or a rope being wound up]

static friction also constrains motion, until it doesn't

(3) Non conservative, non constraint forces

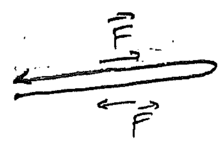
(e.g., viscous or sliding friction)

force depends on direction of motion (usually opposing it)



work done is negative $\quad \vec{F}\cdot d\vec{r} < 0$

Consider closed path



$$\oint \vec{F}\cdot d\vec{r} \neq 0$$

["uphill" both ways]

[joke is that gravity being conservative can't be uphill both ways, but friction can!]

[Return to work energy theorem]

$$dT = \vec{F}\cdot d\vec{r} = \underbrace{\sum_{\text{conserv.}} \vec{F}\cdot d\vec{r}}_{-dU} + \underbrace{\sum_{\text{constr.}} \vec{F}\cdot d\vec{r}}_{0} + \sum_{\text{non cons.}} \vec{F}\cdot d\vec{r}$$

$$dT + dU = \sum_{\text{non cons.}} \vec{F}\cdot d\vec{r}$$

Define mechanical energy

$$E_{mech} = T + U$$

$$dE_{mech} = \sum_{\text{non cons}} \vec{F}\cdot d\vec{r}$$

- negative for dissipative force
- can be positive for driving forces [think driven harmonic oscillator]

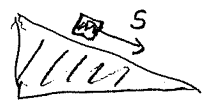
$$\frac{dE_{mech}}{dt} = \sum_{\text{non cons}} \vec{F}\cdot\vec{v} = \text{power delivered by noncons. forces}$$

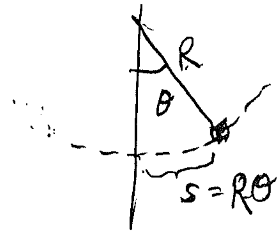
Corollary: if only conservative or constraint* forces act, mechanical energy is conserved

* nonmoving constraints

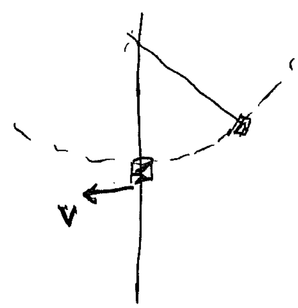
Problem solving w/ constrained forces

1) choose variables that incorporate the constraints

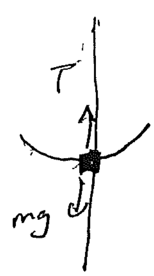




2) use conservation of mechanical energy to determine motion



3) use Newton's law to determine magnitude of constraint force



$$T - mg = m a_z$$

$$T = m\left(g + \frac{v^2}{R}\right)$$

[2 problems]

Angular momentum

For point mass m at $\vec r$ w/ linear momentum $\vec p$

define angular momentum

$$\boxed{\vec L \equiv \vec r \times \vec p}$$

The change in angular momentum

$$\frac{d\vec L}{dt} = \underbrace{\frac{d\vec r}{dt} \times \vec p}_{0 \text{ because } \vec p \parallel \vec v} + \vec r \times \underbrace{\frac{d\vec p}{dt}}_{\vec F}$$

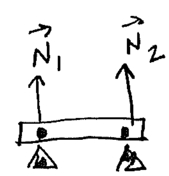
Define torque $\vec\tau$ exerted on a point mass by force $\vec F$

$$\boxed{\vec\tau \equiv \vec r \times \vec F} = \sum_A \vec r \times \vec F^{(A)} = \sum \vec\tau^{(A)}$$

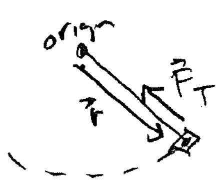
$$\boxed{\frac{d\vec L}{dt} = \vec\tau}$$

$d\vec L = \vec\tau\, dt$ ("angular impulse", twirl acc. More)

Keep in mind that both $\vec L$ and $\vec\tau$
depend on choice of origin
Can eliminate torque of unknown forces.
If place origin at location of one of forces, torque vanishes.



$\vec r \times \vec F_T = 0.$ if $\vec r \parallel \vec F$



Central forces (are directed toward or away from origin) exert no torque

$\Rightarrow$ angular momentum is conserved