Symmetries + conserved quantities

Symmetry ⇒ invariance of Λ [laws of physics] under an operation

⇒ existence of a conserved quantity (Noether's theorem)

In turn, the conserved quantity "generates" the sym. operation

1. time translation ⇒ cons of E
2. spatial translation ⇒ cons of $\vec{p}$
3. rotations ⇒ cons of $\vec{L}$

Invariance under time translation

Suppose Lagrangian has no explicit time dependence

$$\frac{\partial L}{\partial t} = 0$$

L only depends on time thru dependence on configuration variables $q_i(t)$

$$L(q_i(t), \dot{q}_i(t))$$

Consider

$$\frac{dL}{dt} = \sum_i \left(\underbrace{\frac{\partial L}{\partial q_i}}_{= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \text{ by E-L eqns}} \frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right)$$

$$= \sum \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \dot{q}_i \right]$$

$$= \frac{d}{dt}\left(\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$$

$$\Rightarrow \quad 0 = \frac{d}{dt}\left(\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right)$$

$$\Rightarrow \quad \underbrace{\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L}_{H} \text{ is constant in time (conserved)}$$

Consider the infinitesimal time translation of a dynamic variable

$$A(t) \to A(t + dt)$$

$$\delta A = A(t + dt) - A(t)$$

$$= \frac{dA}{dt} dt$$

$$\delta A = dt \{A, H\}$$

we say H generate the time translation (via the Poisson bracket)

Invariance under spatial translation (homogeneity)

Let L describe a system of N particles with positions x_i

Consider a simultaneous translation (shift) of all the particles

$$\begin{cases} x_i \to x_i + a \\ \dot{x}_i \to \dot{x}_i \end{cases}$$

Suppose L is invariant under this shift

$$L(x_i + a, \dot{x}_i) = L(x_i, \dot{x}_i)$$

Consider infinitesimal shift δa

$$L(x_i + \delta a, \dot{x}_i) = L(x_i, \dot{x}_i) + \sum_i \frac{\partial L}{\partial x_i} \delta a + \ldots$$

$$0 = \sum \frac{\partial L}{\partial x_i} = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)$$

$$= \frac{d}{dt}\left(\sum \frac{\partial L}{\partial \dot{x}_i}\right)$$

$$\Rightarrow \sum \frac{\partial L}{\partial \dot{x}_i} = \text{const}$$

But this is just $\sum p_i = p^{sys}$

ie momentum is conserved

Consider spatial translation of a dynamical variable

$$A(x_i) \longrightarrow A(x_i + \delta a)$$

$$\delta A = A(x_i + \delta a) - A(x_i)$$

$$= \sum \frac{\partial A}{\partial x_i} \delta a$$

Consider

$$\{A, p^{sys}\} = \sum_k \left(\frac{\partial A}{\partial x_k} \frac{\partial p^{sys}}{\partial p_k} + \frac{\partial A}{\partial p_k} \underbrace{\frac{\partial p^{sys}}{\partial x_k}}_{0} \right)$$

$$p^{sys} = \sum p_i \quad \text{so} \quad \frac{\partial p^{sys}}{\partial p_k} = 1, \quad \frac{\partial p^{sys}}{\partial x_k} = 0$$

$$= \sum_k \frac{\partial A}{\partial x_k}$$

$$\Rightarrow \quad \delta A = \delta a \{A, p^{sys}\}$$

ie ~~…~~ total momentum generates spatial translations

Invariance under rotations (isotropy)

Under a rotation through angle θ about z-axis, the components of a vector transforms as

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Infinitesimal rotation through $\delta\theta$

$$\begin{pmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x - \delta\theta V_y \\ V_y + \delta\theta V_x \\ V_z \end{pmatrix}$$

$$\Rightarrow \begin{cases} \delta V_x = -\delta\theta \, V_y \\ \delta V_y = \delta\theta \, V_x \\ \delta V_z = 0 \end{cases}$$

Suppose the Lagrangian is invariant under rotations

$$0 = \delta L = L(\vec{x}+\delta\vec{x},\ \dot{\vec{x}}+\delta\dot{\vec{x}}) - L(\vec{x}, \dot{\vec{x}})$$

$$= \frac{\partial L}{\partial \vec{x}} \cdot \delta\vec{x} + \frac{\partial L}{\partial \dot{\vec{x}}} \cdot \delta\dot{\vec{x}}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) \cdot \delta\vec{x} + \frac{\partial L}{\partial \dot{\vec{x}}} \cdot \frac{d}{dt}(\delta\vec{x})$$

$$= \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{\vec{x}}} \cdot \delta\vec{x}}\right)$$

$\vec{p} \cdot \delta\vec{x}$ is conserved

~~But~~ Under a rotation about z-axis

$$\vec{p} \cdot \delta\vec{x} = p_x\,\delta x + p_y\,\delta y$$

$$= p_x(-\delta\theta\, y) + p_y(\delta\theta\, x)$$

$$= \delta\theta\,(x p_y - y p_x)$$

$$= \delta\theta\ L_z$$

Similarly rotations about other axes give other components of $\vec{L}$

Angular momentum generates rotations

$$\delta A = \delta\theta \{A, L_z\}$$

eg consider $A = x$

$$\{A, L_z\} = \{x, xp_y - yp_x\}$$

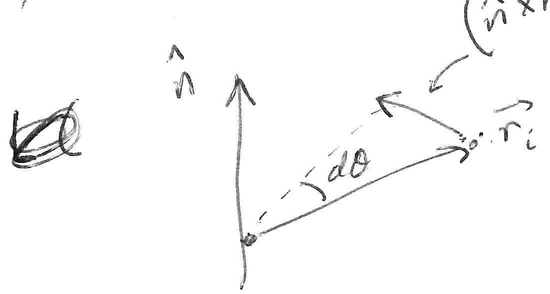
$$= \{x, xp_y\} - \{x, yp_x\}$$

$$= \underbrace{\{x, x\}}_{0} p_y + x \underbrace{\{x, p_y\}}_{0} - \underbrace{\{x, y\}}_{0} p_x - y \underbrace{\{x, p_x\}}_{1}$$

$$= -y$$

$$\Rightarrow \delta x = -y\,\delta\theta \quad \checkmark$$

② Isotropy: Physics is invariant under a rotation of the entire system



$$L(\vec r_i) = L(\vec r_i + \hat n \times \vec r_i \, d\theta, \dot{\vec r}_i + \hat n \times \dot{\vec r}_i \, d\theta)$$

$$= L(\vec r_i) + \underbrace{\sum \frac{\partial L}{\partial \vec r_i} \cdot (\hat n \times \vec r_i) d\theta}_{} + \sum \frac{\partial L}{\partial \dot{\vec r}_i} \cdot (\hat n \times \dot{\vec r}_i) + \dots$$

$$\underbrace{\sum \hat n \cdot \left(\vec r_i \times \frac{\partial L}{\partial \vec r_i}\right) d\theta}_{} + \dots$$

$$\hat n \, d\theta \cdot \sum \left(\vec r_i \times \frac{\partial L}{\partial \vec r_i}\right) + \dot{\vec r}_i \times \frac{\partial L}{\partial \dot{\vec r}_i}$$

$$0 = \sum_i \vec r_i \times \frac{\partial L}{\partial \vec r_i} + \dots = \sum \left(\vec r_i \times \frac{d\vec p_i}{dt} + \frac{d r_i}{dt} \times \vec p_i\right)$$

$$= \frac{d}{dt}\left(\sum \vec r_i \times p_i\right)$$

$$= \frac{d}{dt} \vec L$$

$$\vec L = \text{Const}$$

Rotation invariance $\Rightarrow$ conservation of angular momentum

Virial theorem

$$G = \sum \vec{r}_i \cdot \vec{p}_i$$

$$\frac{dG}{dt} = \sum \dot{\vec{r}}_i \cdot p_i + \sum \vec{r}_i \cdot \dot{\vec{p}}_i$$

$$= \sum \frac{p_i^2}{m_i} + \sum_i \vec{r}_i \cdot \sum_{j \neq i} \vec{F}_{ij}$$

$$= 2T + \sum_{i<j} (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{ij}$$

Now $\vec{F}_{ij} = -k r_{ij}^{n-1} \hat{r}_{ij}$ for an attractive force

associated w/ potential $U = \frac{kr^n}{n}$

$$\frac{dG}{dt} = 2T + \sum_{i<j} (-k r_{ij}^n) = 2T - n \underbrace{\sum_{i<j} U(r_{ij})}_{U}$$

$$\frac{dG}{dt} = 2T - nU$$

$$G(t) - G(0) = \int (2T - nU)\, dt = t\left(2\langle T\rangle - n\langle U\rangle\right)$$

$$2\langle T\rangle - n\langle U\rangle = \frac{G(t) - G(0)}{t}$$

If G is bounded, then rhs $\to 0$ as $t \to \infty$

$$\Rightarrow \boxed{2\langle T\rangle = n\langle U\rangle}$$

$$2\left\langle \sum \tfrac{1}{2} m_i v_i^2 \right\rangle = -\left\langle -\sum_{i<j} \frac{G m_i m_j}{r_{ij}} \right\rangle$$

$$\left\langle \sum m_i v_i^2 \right\rangle = \frac{1}{2}\left\langle \sum_{i\neq j} \frac{G m_i m_j}{r_{ij}} \right\rangle$$

assume N equal masses m

$$N m\langle v^2\rangle = \frac{G N^2 m^2}{2}\left\langle \frac{1}{r}\right\rangle$$

$$\langle v^2\rangle = \frac{G N m}{2}\left\langle \frac{1}{r}\right\rangle$$

But $M_{tot} \approx Nm$

$$\langle v^2\rangle = \frac{G M_{tot}}{2}\left\langle \frac{1}{r}\right\rangle$$

measure avg $\langle v^2\rangle$ + avg $\langle \frac{1}{r}\rangle$ to find $\underline{\underline{M_{tot}}}$.

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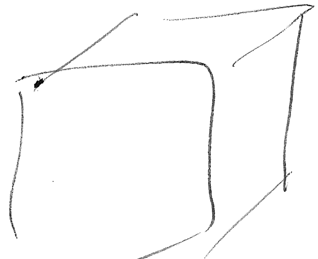
Derivation of ideal gas law



ideal, monatomic

what are forces?

force of constraint.

~~So~~ collision forces cancel out ~~because particles~~

$\vec{F}_{ij} = -\vec{F}_{ji}$

because pcls are at same place $\vec{r}_i = \vec{r}_j$

so $\underbrace{(\vec{r}_i - \vec{r}_j)}_{0 \text{ when } \vec{F}_{ij} \neq 0} \cdot \vec{F}_{ij} = 0$

$$\underbrace{\langle T \rangle}_{\frac{3}{2}NkT} = -\frac{1}{2}\langle \sum \vec{F}_i \cdot \vec{r}_i \rangle$$

$\underbrace{\sum \vec{F}_i}_{-P\, d\vec{A}}$ since $dA \sim \hat{n}$

Force imparted by wall per unit area $= P$

$P\, dA\, dt = \sum F_i\, dt$

$$= \frac{1}{2} P \underbrace{\int d\vec{A} \cdot \vec{r}}_{\int \underbrace{\vec{\nabla} \cdot \vec{r}}_{3}\, dV}$$

$$= \frac{3}{2} PV$$

$PV = NkT$

[what abt diatomic $\frac{5}{2}kT$, $\frac{7}{2}kT$]

(combining T & U)

Rotational coordinates not bounded since no restoring force ...