Poisson brackets

State specified by q_i, p_i

Consider a "dynamical variable" $A(q_i, p_i)$
depends on state of system

~~which depends~~

It may change in time because q_i & p_i change in time, or because it explicitly depends on time

It's time dependence is

$$\frac{dA}{dt} = \sum_i \left(\frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i \right) + \frac{\partial A}{\partial t}$$

use Hamilton's eqns to rewrite

$$= \sum \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

Define the r.h.s ~~as~~ to be ~~the~~ $\{A, H\}$

Poisson bracket between A & H.

More generally the Poisson bracket is

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

Properties of the P.b.

$$\{A, B\} = -\{B, A\} \Rightarrow \{A, A\} = 0$$

$$\{A, BC\} = \{A, B\} C + B \{A, C\}$$ "product rule"

$$\{A, B + C\} = \{A, B\} + \{A, C\}$$ linearity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$$ Jacobi relation

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In QM, we will learn that these properties are shared by the commutator of 2 operators

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{C}$$

This is no surprise because there is a connection between CM + QM via

$$[\hat{A}, \hat{B}] \longleftrightarrow i\hbar \{A, B\}$$

We have $\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$

More specifically

$$\frac{dq_i}{dt} = \{q_i, H\}$$

$$\frac{dp_i}{dt} = \{p_i, H\}$$

These are just Hamilton's eqn's!

$$\{q_i, H\} = \sum_k \Big(\underbrace{\frac{\partial q_i}{\partial q_k}}_{\delta_{ik}} \frac{\partial H}{\partial p_k} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{0} \frac{\partial H}{\partial q_k} \Big) = \frac{\partial H}{\partial p_i}$$

$$\{p_i, H\} = \sum_k \Big(\underbrace{\frac{\partial p_i}{\partial q_k}}_{0} \frac{\partial H}{\partial p_k} - \underbrace{\frac{\partial p_i}{\partial p_k}}_{\delta_{ik}} \frac{\partial H}{\partial q_k} \Big) = -\frac{\partial H}{\partial q_i}$$

Also: if $\{A, H\} = 0$ and if $\frac{\partial A}{\partial t} = 0$ then A is conserved

(ie $\frac{dA}{dt} = 0$)

Note $\{H, H\}$ so H is automatically conserved if $\frac{\partial H}{\partial t} = 0$

(provided it does not depend explicitly on time)

For dynamical variables that depend explicitly on time

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$

Fundamental poisson brackets

$$\{q_i, q_j\} = \sum_k \left(\underbrace{\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k}}_{0} \right) = 0$$

Similarly $\{p_i, p_j\} = 0$

But $\{q_i, p_j\} = \delta_{ij}$

QM analogs

$$[\hat{x}_i, \hat{x}_j] = 0$$

$$[\hat{p}_i, \hat{p}_j] = 0$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$$ ← Heisenberg commutation relation

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In QM: Heisenberg eqn

$$\frac{d\hat{A}}{dt} = \frac{[\hat{A}, \hat{H}]}{i\hbar} + \frac{\partial \hat{A}}{\partial t}$$