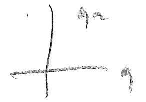
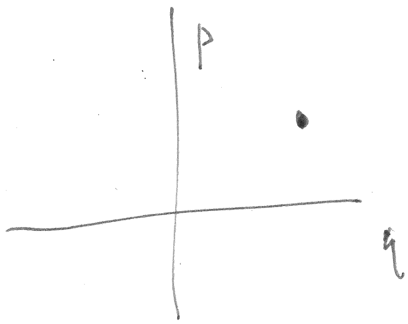
Configuration space (q_i)  need init point + init. velocities

2n-dimensional

Phase space : space parametrized by generalized coords + momenta

e.g. for one degree of freedom, phase space is 2dimensional

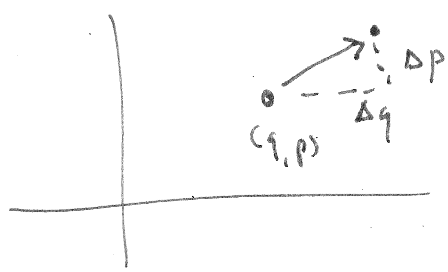


Any state of the system is characterized by a point in phase space.
(give position + velocity at a given time)

As system evolves in time, point ~~moves around~~ follows a trajectory in phase space, according to Hamilton's eqn

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \Rightarrow \quad \Delta q = \frac{\partial H}{\partial p} \Delta t$$

$$\dot{p} = -\frac{\partial H}{\partial p} \qquad \Rightarrow \quad \Delta p = -\frac{\partial H}{\partial q} \Delta t$$



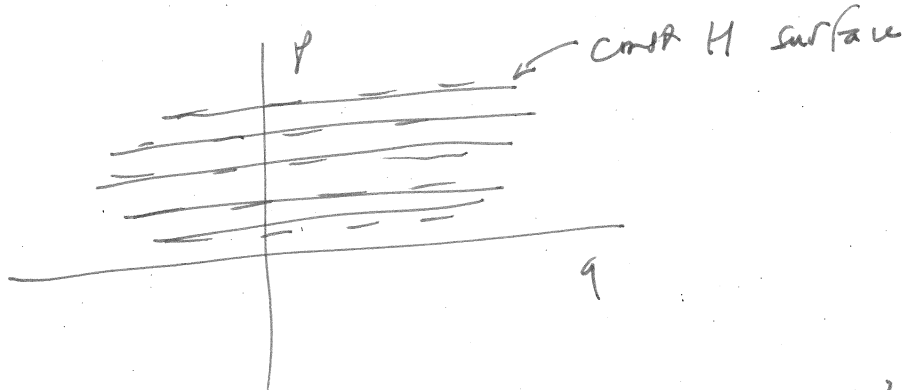
System moves along a vector in phase space

$$v = \begin{pmatrix} \Delta q \\ \Delta p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} \Delta t$$

The Hamiltonian ~~takes on a value~~ is a function on phase space & divides it into surfaces of constant energy (foliates)

(in 2n-dim'l space, the surfaces are (2n−1) dimensional
→ we say they have codimension 1)

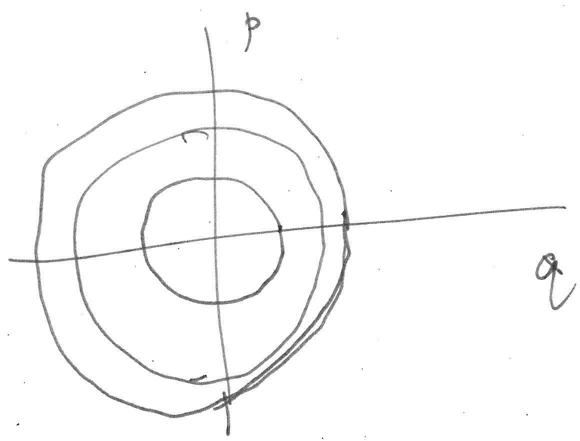
eg for a free particle $H = \frac{p^2}{2m}$



for a harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$

(let $m = 1$, $k = 1$)

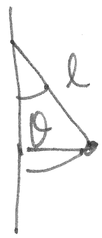
$H = \frac{1}{2}(p^2 + q^2)$



Const H

$p = \pm\sqrt{E - q^2}$

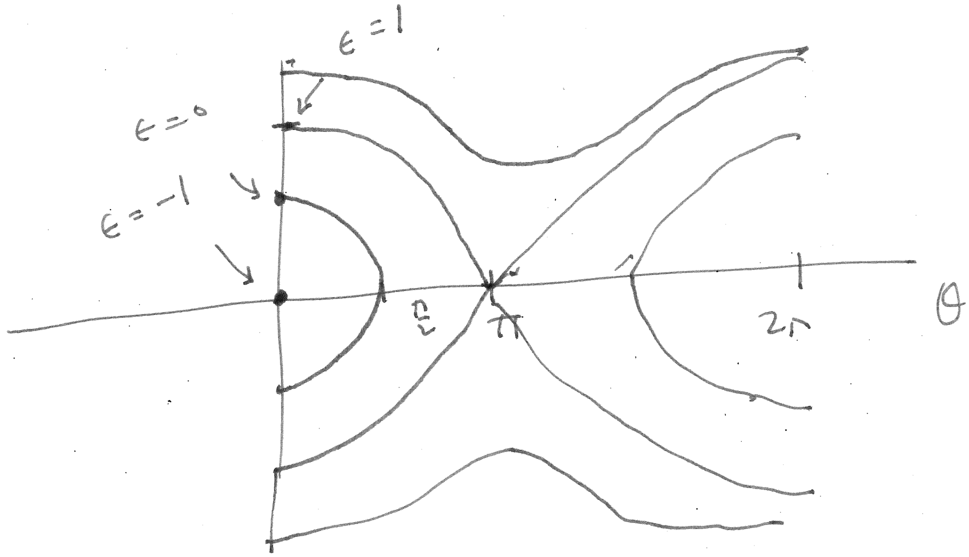
For a planar pendulum $H = \frac{p_\theta^2}{2ml^2} \cancel{\theta} - mgl\cos\theta$



(let $2ml^2 = 1$, $mgl = 1$)

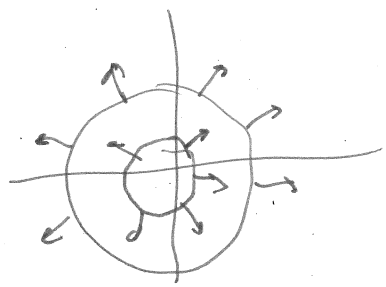
$H = p_\theta^2 - \cos\theta = E$

$p_\theta = \pm\sqrt{E + \cos\theta}$



Gradient of H is a vector field $\perp$ to const energy surfaces

$$\vec{\nabla} H = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$



observe that the direction of flow $\vec{v} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$

is always $\perp$ to the gradient

$$\vec{v} \cdot \vec{\nabla} H = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0$$

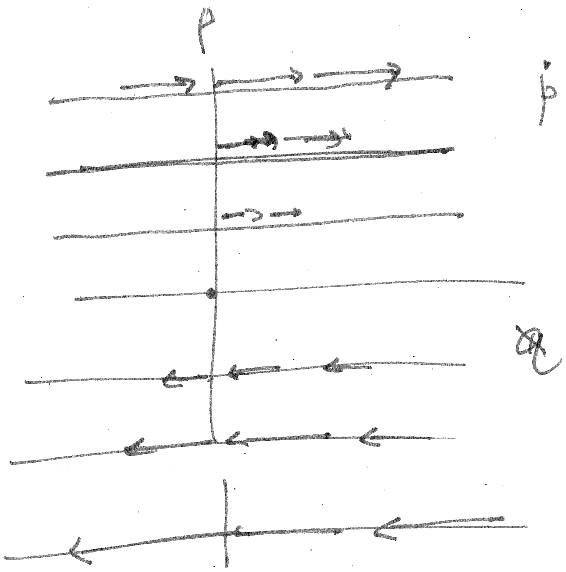
$\therefore$ flow in phase space is always ~~along~~ parallel to const energy surface

(ie energy is conserved as system evolves in time k5

Free particle $\quad H = \frac{p^2}{2m}$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

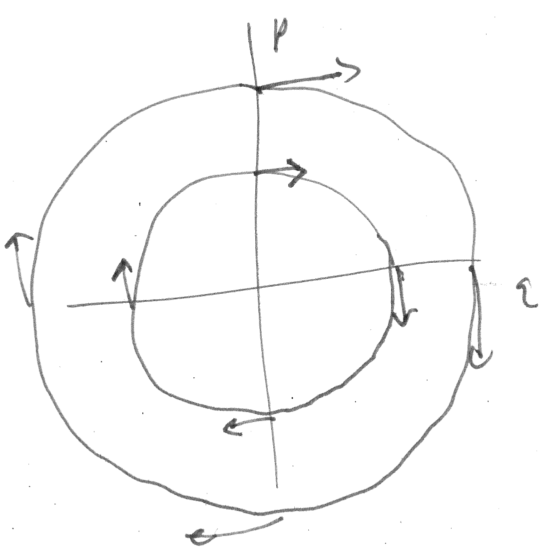
$$\dot{p} = -\frac{\partial H}{\partial q} = 0$$



Harmonic oscillator $\quad H = \frac{1}{2}(p^2 + q^2)$

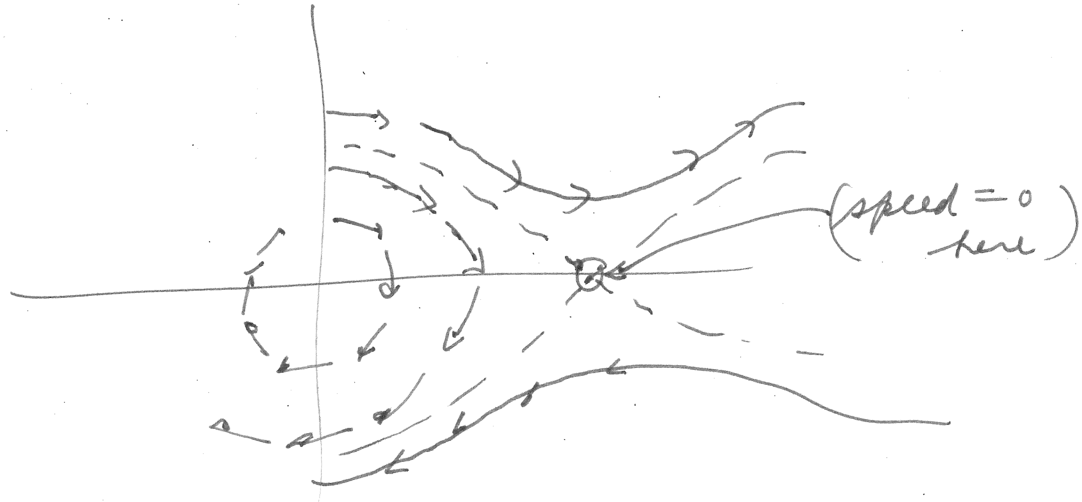
$$\dot{q} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -q$$

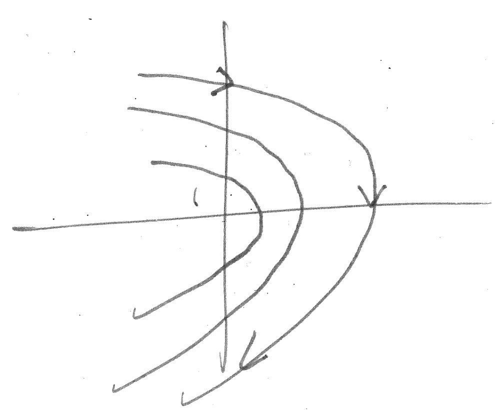


observe: time to flow around circle indep. of radius!

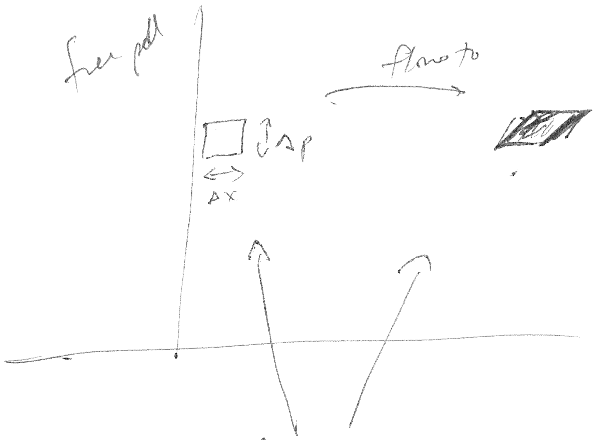
Pendulum

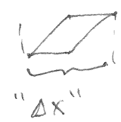


[do particle in const grav field as problem]



Consider a region of phase space
reflecting uncertainty in
initial conditions



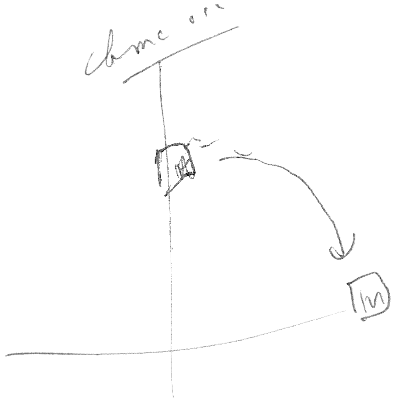


~~volume~~ area of region remains constant (Liouville's thm)

(flows as an incompressible fluid

because $\vec{\nabla}\cdot\vec{v} = \cancel{\frac{\partial}{\partial q}(}\;\frac{\partial v_q}{\partial q} + \frac{\partial v_p}{\partial p} =$

$$= \frac{\partial}{\partial q}\left(+\frac{\partial H}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0$$



Spencer asked: if $\vec{\nabla}\cdot\vec{v}=0$ is $\vec{v}=\vec{\nabla}\times(\text{vector field})$?

$$\vec{v}=\begin{pmatrix}\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial q}\end{pmatrix}=\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix}$$

Let $\vec{H}=(0,0,H)$

$$\vec{v}=\vec{\nabla}\times\vec{H}=\begin{pmatrix}\hat{q} & \hat{p} & \hat{z}\\ \frac{\partial}{\partial q} & \frac{\partial}{\partial p} & \frac{\partial}{\partial z}\\ 0 & 0 & H\end{pmatrix}=\begin{pmatrix}\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial q}\\ 0\end{pmatrix}$$

$$\vec{\nabla}\cdot\vec{v}=0$$

$$\vec{\nabla}H=\begin{pmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{pmatrix}\implies\vec{\nabla}\times(\vec{\nabla}H)=0$$

$$\vec{v}=\begin{pmatrix}\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial q}\end{pmatrix}\implies\vec{\nabla}\cdot\vec{v}=0$$

but $\vec{\nabla}\times\vec{v}\sim\nabla^2H\neq0$