Hamiltonian formulation of mechanics

William Rowan Hamilton (cf Tim Blais)

start w/ Lagrangian

$L(q_i, \dot{q}_i, t)$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{momentum conjugate to } q_i$$

Imagine inverting this eqn to obtain $\dot{q}_i$ in terms of p_i and q_i

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eg. ~~pot~~ central potential

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r)$$

$$\begin{cases} p_r = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \dfrac{p_r}{m} \\ p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \dfrac{p_\theta}{mr^2} \end{cases}$$

Having obtained $\dot{q}_i(q, p, t)$

define the Hamiltonian

$$H(q,p,t) \equiv \sum_i p_i \, \dot{q}_i(q,p,t) - L\left(q, \dot{q}(q,p,t), t\right)$$

expressed in terms of q, p, and t.

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e.g. $L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r,\theta)$

$$\Rightarrow H = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{m} - \left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - U(r,\theta) \right)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r,\theta)$$

In taking partial derivatives of $L(q, \dot{q}, t)$

$\frac{\partial L}{\partial q}$ means hold $\dot{q}$ fixed

$\frac{\partial L}{\partial \dot{q}}$ means hold q fixed

But in taking partial derivatives of $H(q, p, t)$

$\frac{\partial H}{\partial q}$ means hold p fixed

$\frac{\partial H}{\partial p}$ means hold q fixed

$\dot{q}$ is considered a function of $q + p$ $\Rightarrow$ use chain rule

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$$H = p\,\dot{q}(q, p, t) - L(q, \dot{q}(q, p, t), t)$$

$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p,\ \text{by def}}\frac{\partial \dot{q}}{\partial q} = -\frac{\partial L}{\partial q} \underset{\text{by E-L}}{=} -\dot{p}$$

~~use E-L to get~~

$$\frac{\partial H}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p,\ \text{by def}}\frac{\partial \dot{q}}{\partial p} = \dot{q}$$

$$\frac{\partial H}{\partial t} = p\frac{\partial \dot{q}}{\partial t} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\Rightarrow \left.\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}\right\} \text{Hamilton's equations or "the canonical equations"}$$

more generally, for N degrees of freedom q_i, $i=1,\dots N$

$$\left.\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}\\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned}\right\} \begin{array}{l}2N\\ 1^{st}\text{ order eqns}\end{array} \quad \text{vs} \quad \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}_{\text{Euler-Lagrange}} \left.\right\} \begin{array}{l}N\\ 2^{nd}\\ \text{order}\\ \text{eqns}\end{array}$$

Hamilton $\longleftarrow$ equivalent $\longrightarrow$

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eg $H = \dfrac{p_r^2}{2m} + \dfrac{p_\phi^2}{2mr^2} + U(r)$

$$\left.\begin{aligned}\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}\\ \dot{p}_r &= -\frac{\partial H}{\partial r} = -\frac{\partial U}{\partial r} + \frac{p_\phi^2}{mr^3}\end{aligned}\right\} \Rightarrow m\ddot{r} = \underbrace{\frac{p_\phi^2}{mr^3}}_{mr\dot{\phi}^2} - \frac{\partial U}{\partial r}$$

$$\left.\begin{aligned}\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}\\ \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0\end{aligned}\right\} \Rightarrow p_\phi = mr^2\dot{\phi} = \text{const}$$

we can set that momentum equal to a constant l & henceforth ignore it

$$H = \frac{p_r^2}{2m} + \underbrace{\frac{l^2}{2mr^2} + U(r)}_{U_{eff}(r)}$$ $\longleftarrow$ Treat this as Hamiltonian w/ only one coordinate r

~~one dimension~~

ϕ is called an ignorable coordinate

Whenever the Hamiltonian doesn't depend on a ~~particle~~ particular coordinate (e.g. ϕ) that coordinate is called "cyclic" and the corresponding momentum is a constant

because $\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0$

We can set it to a const in H from the outset

q_i is also called "ignorable"

(In Lagrangian it isn't ignorable because even if $\frac{\partial L}{\partial \phi} = 0$, $\dot{\phi}$ is not constant)

K4

h conserved if $\frac{\partial L}{\partial t} = 0$,

Does $h = T + V$?

Yes, if ⓐ V is velocity independent [not true if magnetic field]

ⓑ constraints are scleronomic,

ie relations between Cartesian + generalized coords do <u>not</u> depend on time

$x_i = x_i(q, \cancel{t})$

ⓐ $\Rightarrow \frac{\partial V}{\partial \dot{q}_i} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$

ⓑ $\Rightarrow T = a_{jk}(q)\, \dot{q}_j \dot{q}_k + \underbrace{\cancel{b_i \dot{q}_i}}_{0} + \underbrace{\cancel{c}}_{0}$ since $\frac{\partial x_i}{\partial t} = 0$

$$\frac{\partial T}{\partial \dot{q}_i} = a_{jk} \dot{q}_j \underbrace{\frac{\partial \dot{q}_k}{\partial \dot{q}_i}}_{\delta_{ki}} + a_{jk} \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{q}_i}}_{\delta_{ji}} \dot{q}_k$$

$$= a_{ji} \dot{q}_j + a_{ik} \dot{q}_k$$

$$= 2 a_{ij} \dot{q}_j$$

$$\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2 a_{ij} \dot{q}_i \dot{q}_j = 2T$$

$$h = 2T - L = T + V \quad ✓$$