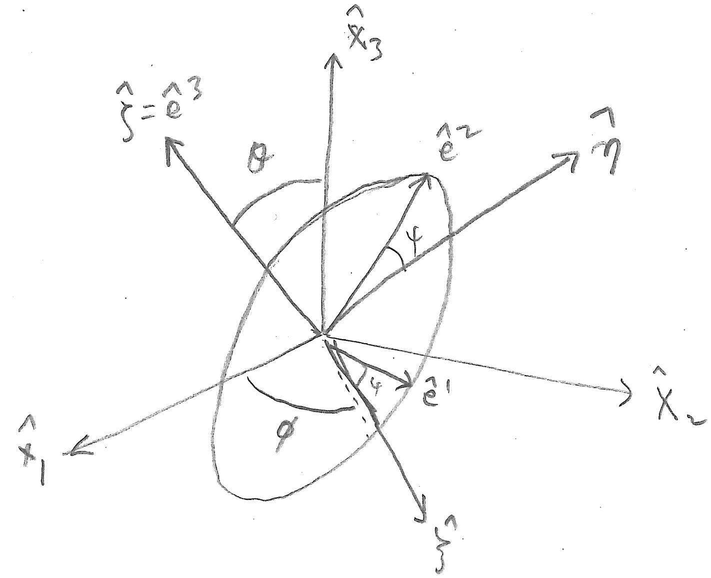
Rigid body motion from a Lagrangian

Review Euler angles

3 d.o.f.



$$\vec{\omega} = \underset{\substack{\text{spinning}\\ \text{around } \hat{\zeta}}}{\dot{\psi}\hat{\zeta}} + \underset{\substack{\text{precessing}\\ \text{around } \hat{x}_3}}{\dot{\phi}\hat{x}_3} + \underset{\substack{\text{tipping}\\ \text{about } \hat{\xi}}}{\dot{\theta}\hat{\xi}}$$

$\dot{\psi}$ = spin rate

$\dot{\phi}$ = precession rate

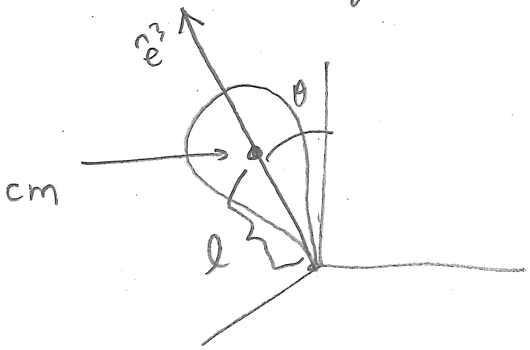
$$\hat{x}_3 = \hat{\zeta}\cos\theta + \hat{\eta}\sin\theta$$

$$\vec{\omega} = \underbrace{(\dot{\psi} + \dot{\phi}\cos\theta)}_{\omega_3}\hat{\zeta} + \underbrace{(\dot{\phi}\sin\theta)}_{\omega_\eta}\hat{\eta} + \underbrace{\dot{\theta}}_{\omega_\xi}\hat{\xi}$$

$$\left\{\begin{aligned} \omega_1 &= \omega_\xi\cos\psi + \omega_\eta\sin\psi = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ \omega_2 &= -\omega_\xi\sin\psi + \omega_\eta\cos\psi = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}\right.$$

components wrt. body fixed axes.

Consider a symmetric top with tip fixed
(two equal principal moments)



$$I_1 = I_2 = I_\perp$$

ℓ = distance from fixed tip to cm

$$T = \frac{1}{2} \sum_k I_k \omega_k^2$$

$$= \frac{1}{2} I_\perp (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

$$= \frac{1}{2} I_\perp (\dot\theta^2 + \dot\phi^2 \sin^2(\theta)) + \frac{1}{2} I_3 (\dot\psi + \dot\phi \cos\theta)^2$$

$$U = mgh = mg\ell \cos\theta$$

$$L = T - U$$

Note that L is independent of ϕ and ψ (cyclic coords)

$\Rightarrow \exists$ 2 conserved quantities: $p = p_\phi$ and $p' = p_\psi$

Also L indep of t so E also conserved

$$p_\psi = \cancel{p_\psi} = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = p \text{ const}, \text{ because } \frac{\partial L}{\partial \psi} = 0$$

$$p_\phi\ \cancel{p_\phi} = \frac{\partial L}{\partial \dot\phi} = I_1\dot\phi\sin^2\theta + \underbrace{I_3(\dot\psi + \dot\phi\cos\theta)}_{p}\cos\theta = p' \text{ const}, \ \frac{\partial L}{\partial \phi} = 0$$

Rewrite $\dot\phi$ and $\dot\psi$ in terms of p_ψ, p_ϕ, θ

$$\boxed{\dot\phi = \frac{p' - p\cos\theta}{I_1\sin^2\theta}}$$

$$\dot\psi = \frac{p}{I_3} - \dot\phi\cos\theta$$

$$\boxed{\dot\psi = \frac{p}{I_3} - \frac{(p' - p\cos\theta)\cos\theta}{I_1\sin^2\theta}}$$

$$p'' = \frac{\partial L}{\partial \dot\theta} = I_1\dot\theta$$

Since L depends on θ, $\frac{\partial L}{\partial \dot\theta}$ is not conserved

$$\frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta}$$

What is the physical meaning of ~~p_ψ, p_ϕ, p_θ~~ p, p', p''?

$\hat{e}_3$ = principle axis

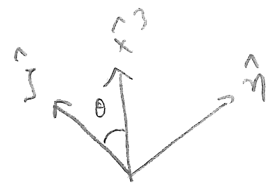
$$p_\psi = I_3(\dot\psi + \dot\phi \cos\theta) = I_3 \omega_3 = L_3$$

$p_\psi = L_3$ = component of angular momentum along body axis $\hat{e}^3 \equiv \hat{\zeta}$

$$p_\phi = I_\perp \dot\phi \sin^2\theta + p_\psi \cos\theta$$

$$= I_\perp \omega_\eta \sin\theta + I_3 \omega_3 \cos\theta$$

$\hat{\eta}$ + $\hat{e}_3$ are principle axes



$$= L_\eta \sin\theta + L_\zeta \cos\theta$$

$$= L_3$$

$p_\phi = L_3$ = component of angular momentum along space fixed axis $\hat{x}_3$

$\hat{\xi}$ = principle axis

$$p_\theta = I_\perp \dot{\theta} = I_\perp \omega_\xi = L_\xi$$

= component of angular momentum along $\hat{\xi}$ axis

→ not conserved in general

In particular $$\frac{dp_\theta}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

Consider also

L_η = component of angular momentum along $\hat{\eta}$ axis

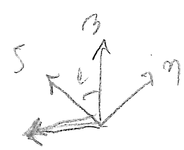
$$= I_\perp \omega_\eta = I_\perp \dot{\phi} \sin\theta$$

③ Why L_ζ cons'd & L_η not? Or is it?

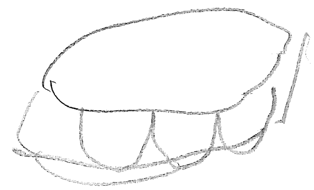
$$L_3 = L_\eta \sin\theta + L_\zeta \cos\theta$$

$$0 = \frac{dL_\eta}{dt}\sin\theta + L_\eta \cos\theta\,\dot\theta + \cancel{\frac{dL_\zeta}{dt}\cos\theta} - L_\zeta \sin\theta\,\dot\theta$$

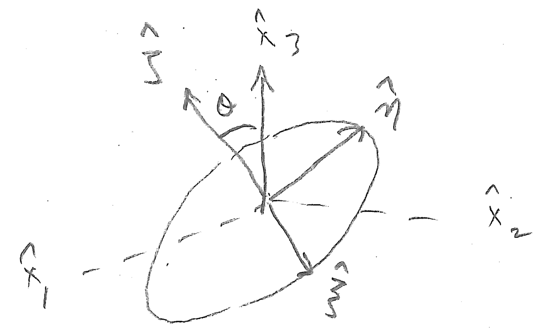
$$\frac{dL_\eta}{dt}\sin\theta + (L_\eta\cos\theta - L_\zeta\sin\theta)\,\dot\theta$$



Problem on nutation?



Revisit torque-free motion of symmetrical object ($I_1 = I_2$)



$\hat{\zeta}, \hat{\xi}, \hat{\eta}$ are all principal axes

In terms of Euler angles

$$L_\zeta = I_3 \omega_3 = I_3(\dot{\psi} + \dot{\phi}\cos\theta)$$

$$L_\eta = I_\perp \omega_\eta = I_\perp \dot{\phi} \sin\theta$$

$$L_\xi = I_\perp \omega_\xi = I_\perp \dot{\theta}$$

External torque $= 0 \Rightarrow \vec{L} =$ constant

For convenience, choose $\vec{L}$ along $\hat{x}_3$ space axis

$$\vec{L} = L\hat{x}_3$$

$$\Rightarrow \begin{cases} L_\zeta = L\cos\theta \\ L_\eta = L\sin\theta \\ L_\xi = 0 \end{cases}$$

$$L = \sqrt{L_\zeta^2 + L_\eta^2} = \sqrt{I_3^2\omega_3^2 + I_\perp^2\omega_\perp^2} = \omega_3\sqrt{I_3^2 + I_\perp^2 \tan^2\alpha}$$

Compare L_ξ equations $\Rightarrow \dot{\theta} = 0 \Rightarrow \theta =$ const

(as we learned earlier angle between $\hat{x}_3$ + $\hat{e}^3$ is const)

Compare L_η equations $\Rightarrow \dot{\phi} = \dfrac{L}{I_\perp} =$ const

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Compare L_ζ equations $\Rightarrow \dot{\psi} = \dfrac{L}{I_3}\cos\theta - \dot{\phi}\cos\theta = \left(\dfrac{L}{I_3} - \dfrac{L}{I_\perp}\right)\cos\theta$

$$= \frac{I_\perp - I_3}{I_3 I_\perp} L\cos\theta = \left(\frac{I_\perp - I_3}{I_\perp}\right)\frac{L_\zeta}{I_3} = \left(\frac{I_\perp - I_3}{I_\perp}\right)\omega_3 = -\Omega$$

(as before).

Symmetrical top subject to gravity

Recall $L = \frac{1}{2} I_\perp (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - mgl\cos\theta$

Also recall conserved conjugate momenta.

$$p = \frac{\partial L}{\partial \dot\psi} = I_3 (\dot\psi + \dot\phi\cos\theta) = \text{const}$$

$$p' = \frac{\partial L}{\partial \dot\phi} = I_\perp \dot\phi \sin^2\theta + I_3 (\dot\psi + \dot\phi\cos\theta)\cos\theta = \text{const}$$

Invert these to get

$$\dot\phi = \frac{p' - p\cos\theta}{I_\perp \sin^2\theta}$$

$$\dot\psi = \frac{p}{I_3} - \frac{(p' - p\cos\theta)\cos\theta}{I_\perp \sin^2\theta}$$

E-L eqn for θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{\partial L}{\partial\theta}$$

$$I_\perp \ddot\theta = I_\perp \dot\phi^2 \sin\theta\cos\theta - I_3(\dot\psi + \dot\phi\cos\theta)\dot\phi\sin\theta + mgl\sin\theta$$

$$(*) \qquad I_\perp \ddot\theta = I_\perp \sin\theta\cos\theta\,\dot\phi^2 - p\sin\theta\,\dot\phi + mgl\sin\theta$$

First consider solutions w/ $\theta = \theta_0$ (const)

i.e. top w/ fixed tilt angle

p_ψ, p_ϕ', θ const $\Rightarrow$ $\dot\phi$ and $\dot\psi$ const

steady spin + precession rates

The precession rate is determined by (*)

$$\ddot\theta = 0 \Rightarrow 0 = I_\perp \cos\theta_0 \, \dot\phi^2 - p_\psi \, \dot\phi + mgl$$

$$\dot\phi = \frac{p_\psi \pm \sqrt{p_\psi^2 - 4mgl\, I_\perp \cos\theta_0}}{2 I_\perp \cos\theta_0}$$ $\leftarrow$ 2 possible precession rates (for given spin rate)

$$= \frac{p_\psi}{2 I_\perp \cos\theta_0}\left[1 \pm \underbrace{\sqrt{1 - \frac{4mgl\, I_\perp}{p_\psi^2}\cos\theta_0}}_{\approx 1 - \frac{2mgl\, I_\perp}{p_\psi^2}\cos\theta_0} \right]$$

$\approx 1 - \frac{2mgl\, I_\perp}{p_\psi^2}\cos\theta_0$ for p_ψ large (a fast spinning top)

$$\dot\phi \approx \begin{cases} \dfrac{p_\psi}{I_\perp \cos\theta_0} & \text{fast precession (comparable to spin rate)} \\ & \hookrightarrow \text{hard to set up} \\ \dfrac{mgl}{p_\psi} & \text{slow precession} \end{cases}$$

For slow precession, $p_\psi = I_3(\dot\psi + \cancel{\dot\phi \cos\theta})$ $\rightarrow$ neglect

$$\Rightarrow \dot\phi \approx \frac{mgl}{I_3 \dot\psi}$$ as we found earlier in the course

($\dot\phi$ indep of θ)

Next consider $\theta \neq$ const.

$$(*) \quad I_\perp \ddot{\theta} = I_\perp \sin\theta \cos\theta \, \dot{\phi}^2 - p \sin\theta \, \dot{\phi} + mgl \sin\theta$$

But $\dot{\phi} = \dfrac{p' - p\cos\theta}{I_\perp \sin^2\theta}$ can be used to eliminate $\dot{\phi}$

2nd order ode for θ

$$I_\perp \ddot{\theta} = \underbrace{\frac{(p' - p\cos\theta)^2 \cos\theta}{I_\perp \sin^3\theta} - \frac{p\,(p' - p\cos\theta)}{I_\perp \sin\theta} + mgl\sin\theta}$$

1st 2 terms equal

$$\left[\frac{(p' - p\cos\theta)(p'\cos\theta - p)}{I_\perp \sin^3\theta} \right]$$

$$= -\frac{d}{d\theta}\Bigg[\underbrace{\frac{(p' - p\cos\theta)^2}{2 I_\perp \sin^2\theta} + mgl\cos\theta}_{U^{eff}(\theta)} \Bigg]$$

[How did we know? L indep of $t \Rightarrow$ energy conservation]

As usual

$$I_\perp \ddot{\theta}\dot{\theta} = -\frac{dU^{eff}}{d\theta}\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{I_\perp \dot{\theta}^2}{2}\right) = -\frac{d}{dt}U^{eff}$$

$$\frac{1}{2} I_\perp \dot{\theta}^2 + U^{eff} = E$$

3 constants (E, p, p') determine the top's motion

$$U^{eff} = \frac{(p' - p\cos\theta)^2}{2I\sin^2\theta} + mgl\cos\theta$$

$$\frac{dU^{eff}}{d\theta} = -\frac{(p'-p\cos\theta)^2\cos\theta}{I\sin^3\theta} + \frac{(p'-p\cos\theta)\,p\sin\theta}{I\sin^2\theta} - mgl\sin\theta$$

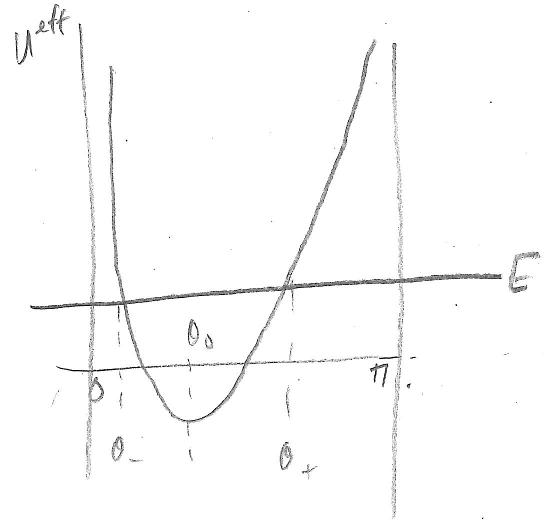
$$= \frac{(p'-p\cos\theta)\left[-(p'-p\cos\theta)\cos\theta + p\sin^2\theta\right]}{I\sin^3\theta} - mgl\sin\theta$$

$$= -\frac{(p'-p\cos\theta)(p'\cos\theta - p)}{I\sin^3\theta} - mgl\sin\theta$$

$$\frac{d^2U^{eff}}{d\theta^2} = \frac{-p(p'\cos\theta - p)\sin^2\theta + (p'-p\cos\theta)\cdot p'\sin^2\theta + 3(p'-p\cos\theta)(p'\cos\theta-p)\cos\theta}{I\sin^4\theta} - mgl\cos\theta$$

$$= \frac{p^2(\sin^2\theta + 3\cos^2\theta) + pp'(-2\cos\theta\sin^2\theta - 3\cos\theta - 3\cos^3\theta) + p'^2(\sin^2\theta + 3\cos^2\theta)}{I\sin^4\theta} - mgl\cos\theta$$

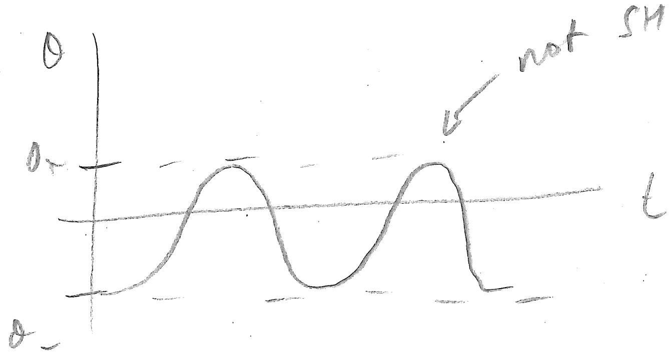
$$= \frac{p^2(1+2\cos^2\theta) + pp'(-5 - \cos^2\theta)\cos\theta + p'^2(1+2\cos^2\theta)}{I\sin^4\theta} - mgl\cos\theta$$



top's tilt ranges between θ_- + θ_+

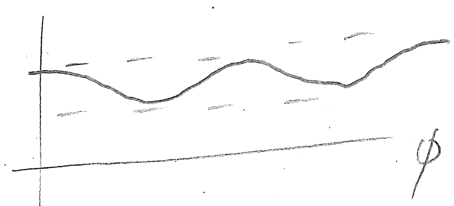
unless amplitude small

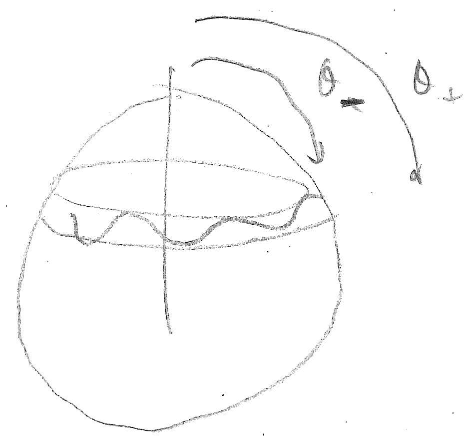
not SHM, but periodic



nutation ("nodding")

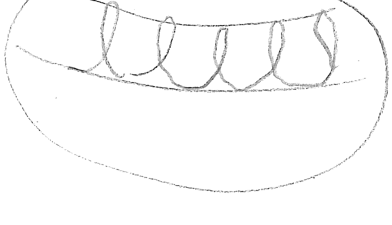
For small amplitude, $\dot{\phi} \simeq$ const so

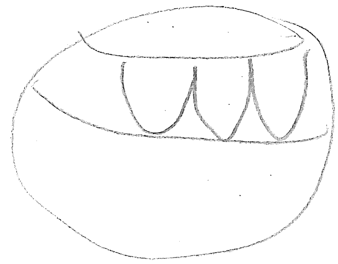




For large amplitude, $\dot{\phi}$ varies, + can even change sign

if $\dot{\phi} = 0$ at θ_-





typical

(Youtube: gyroscope precession + nutation)

Top problem

vertical top: 10.56 Taylor
p. 120, prob 2 of Slater

starting $\theta(0)=\theta_0$ and $\dot\phi(0)=0$ $\Rightarrow$ $p_\phi = p_\psi \cos\theta_0$

find subsequent motion

steady precession (p. 120, prob 3 of Slater)

simple oscillation abt steady precession

$\rightarrow$ freq. of nutation

if $\theta < \frac{\pi}{2}$!

min $\dot\psi$ for which steady precession can occur

(discrim = 0)

$\rightarrow p_\psi^2 > 4mgl I_1 \cos\theta_0$

find $\dot\phi$ as fcn of $\dot\psi$

Fast precession: exact soln for $\dot\psi$ & $\dot\phi$

in that case, but nevertheless if we keep only the linear term we shall still find that θ is governed by the equation of simple harmonic motion, so that θ will still oscillate about the fixed value given by (4.10). The general motion, combining an oscillation of θ with a precession of the axis of figure, is called "nutation." Here as before, when we have solved for θ as a function of time, we can determine φ and ψ by (4.4). Thus we have in effect a general solution of the problem of the top spinning under gravity, and a general description of its motion.

Problems

1. Discuss the motion of a symmetrical top under gravity by the energy method, showing that the effective potential energy can be written in the alternative forms

$$V' = mgL\cos\theta + \frac{(p_\varphi^2 + p_\psi^2 - 2p_\varphi p_\psi \cos\theta)}{2I_1 \sin^2\theta} + \text{constant}$$
$$= mgL\cos\theta + \frac{1}{2}I_3\omega_3^2 + \frac{(p_z - I_3\omega_3\cos\theta)^2}{2I_1\sin^2\theta}.$$

2. A top is started spinning vertically, with no other motion, so that initially $\theta = 0$, $d\theta/dt = 0$. Show that $p_z = I_3\omega_3$, E = energy = $\frac{1}{2}I_3\omega_3^2 + mgL$. Substituting these expressions in the energy equation $E = \frac{1}{2}I_1\dot{\theta}^2 + V'$, show that, if $\omega_3 > \omega'$, where $(\omega')^2 = 4mgL(I_1/I_3^2)$, the angle θ must remain equal to zero, but that if ω_3 falls below ω', θ will oscillate between 0 and the angle $\cos^{-1}[2(\omega_3/\omega')^2 - 1]$. Experimentally, if a top is started as we have described, with $\omega_3 > \omega'$, there will be a frictional torque decreasing ω_3, and as soon as the torque reduces ω_3 below ω', the top will begin to wobble.

3. For a nutation of small amplitude about the steady precessional motion of a top, the angle θ oscillates sinusoidally about the equilibrium angle. Find the frequency of the nutation, by expanding the potential V' in power series in $\theta - \theta_0$, where θ_0 is the angle of steady precession with the same angular momentum. Retain only the constant and the term in $(\theta - \theta_0)^2$, and get the frequency by comparing with the corresponding expression for the linear oscillator.

4. In Fig. 19, show that the tangent of the angle between ω and the axis of figure is a/ω_3, and the tangent of the angle between the z axis and the ζ axis is $(I_1/I_3)(a/\omega_3)$, where a, ω_3 represent the components of the angular velocity at right angles to and along the figure axis. Knowing from (3.5) that the time required for the axis of angular velocity to perform a complete rotation with respect to the body is $\tau = (2\pi/\omega_3)[I_1/(I_1 - I_3)]$, show that the time for it to perform a complete rotation in space is approximately $(2\pi/\omega_3)(I_1/I_3)$, if the angles mentioned above are small. Hence show that for the earth the axis of angular velocity is not fixed, but rotates about a fixed direction approximately once a day.

5. The earth is acted on by torques exerted by the sun and moon, and as a consequence its angular momentum precesses about a fixed direction in space. This is entirely separate from the effect of Fig. 19 and Prob. 4, which we now neglect. This precession has a period of 25,800 years, and carries the angular momentum about a cone of semivertical angle 23°27′, so that the pole in succession points to different parts of the heavens, resulting in the precession of the equinoxes, and in the fact that different stars act as pole star at different periods of history. Show that the motion can be represented by the rolling of a cone fixed in the earth, of diameter 21 in. at the north pole, on a cone of angle 23°27′ fixed in the heavens.

6. A system of electrons moving about a center of attraction has a certain angular momentum, equal to $\Sigma m(\mathbf{r} \times \mathbf{v})$, and also a magnetic moment, equal to $\frac{1}{2}\Sigma e(\mathbf{r} \times \mathbf{v})$, where e is the charge and m the mass of an electron. This magnetic effect results because the electrons in rotation act like little currents, which in turn have magnetic fields like bar magnets. An external magnetic induction $\mathbf{B}$ exerts a torque on the system, equal to the vector product of the magnetic moment and $\mathbf{B}$. Show that, under the action of the field, the system of electrons precesses with angular velocity $e\mathbf{B}/2m$ about the direction of the field. This precession, which, as we see, is independent of the velocities of the electrons, is called "Larmor's precession."