Orbit equation

We've studied $r(t)$ and $\phi(t)$.

Let's see if we can find the eqn for the orbit $r(\phi)$

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \frac{\ell}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}(E-U_{eff}(r))}}$$

$$\phi(r) - \phi_0 = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2\mu}{\ell^2}(E-U_{eff}(r))}}$$

Solve analytically or numerically then invert to get $r(\phi)$. ← see old 370 notes for soln for inverse square

---

If $U(r) = \frac{kr^n}{n}$ then can be done in terms of

simple functions if $n = 2, -1, -2$
($n=2$: Hooke; $n=-1$: inverse square)

and in terms of elliptic functions for several other values.

($n = 6, 4, 1, -3, -4, -6$)

Try another approach.

Euler-Lagrange gives $\frac{d\phi}{dt} = \frac{\ell}{\mu r^2}$

and $\mu \frac{d^2 r}{dt^2} = \frac{\ell^2}{\mu r^3} + F_r$ where $F_r = -\frac{dU}{dr}$

Try writing the o.d.e w/ independent variable ϕ instead of t

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}$$

Also try using $u = \frac{1}{r}$ for dependent variable

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{\ell}{\mu r^2}\frac{du}{d\phi} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = \frac{\ell}{\mu r^2}\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2}{\mu^2}u^2\frac{d^2 u}{d\phi^2}$$

$$\Rightarrow -\frac{\ell^2}{\mu}u\frac{d^2 u}{d\phi^2} = \frac{\ell^2}{\mu}u^3 = F_r\left(\frac{1}{u}\right)$$

$$\frac{d^2 u}{d\phi^2} + u = -\frac{\mu}{\ell^2 u^2}F_r\left(\frac{1}{u}\right)$$

$\rightarrow$ do free pcl

(Note if we know the orbit $r(\phi)$ we can use this to compute the force.)

h2.5

<u>Free particle</u> $F_r = 0$

$$\frac{d^2u}{d\phi^2} + u = 0$$

$$u = A\cos(\phi - \phi_0)$$

$$\frac{1}{r} = A\cos(\phi - \phi_0)$$

$$r\cos(\phi - \phi_0) = \text{const}$$

straight line

do this

<u>Harm osc</u> $F_r = -kr = -\frac{k}{u}$

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu k}{l^2 u^3}$$

Not so obvious

$$\phi(r) - \phi_0 = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2\mu}{l^2}\left(E - \frac{1}{2}kr^2\right)}}$$

Specialize to <u>inverse square law</u>

$$F_r = -\frac{k}{r^2} \qquad k = Gm_1m_2 \text{ for gravity}$$

$$= \frac{e^2}{4\pi\epsilon_0} \text{ for hydrogen atom}$$

$$= -ku^2$$

$\Rightarrow$ orbit eqn $\boxed{\frac{d^2u}{d\phi^2} + u = \frac{k\mu}{\ell^2}}$

$$u_p = \frac{k\mu}{\ell^2}$$

$$u_c = A\cos(\phi - \phi_0)$$

$$u = \frac{k\mu}{\ell^2} + A\cos(\phi - \phi_0)$$

$$= \frac{k\mu}{\ell^2}\left[1 + e\cos(\phi - \phi_0)\right]$$

e = eccentricity

$$\boxed{r = \frac{\ell^2}{\mu k\left[1 + e\cos(\phi - \phi_0)\right]}}$$

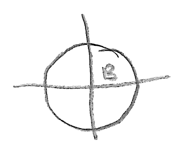
For simplicity, let $\phi_0 = 0$ (just reorient the coords.)

Define $B = \dfrac{\ell^2}{\mu k}$

$$\Rightarrow \boxed{r = \frac{B}{1 + e\cos\phi}}$$ eqn for conic section (2 parameters: size B, eccentricity e)

Can choose $e \geq 0$. (If $e < 0$, choose $\phi_0 = \pi$ to change the sign.)

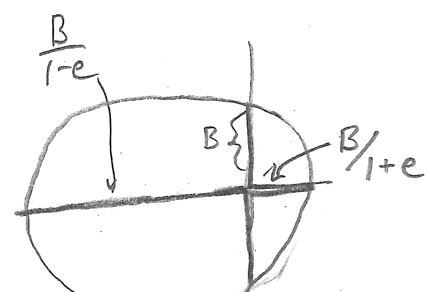
$e = 0$ (circle) $r = B$



$0 < e < 1$ (ellipse) $\phi = 0 \Rightarrow r_- = \dfrac{B}{1+e}$

$\phi = \pi \Rightarrow r_+ = \dfrac{B}{1-e}$

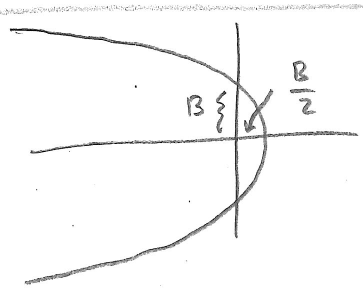
$\phi = \frac{\pi}{2} \Rightarrow r = B$



$e = 1$ (parabola) $\phi = 0 \Rightarrow r_- = \frac{B}{2}$

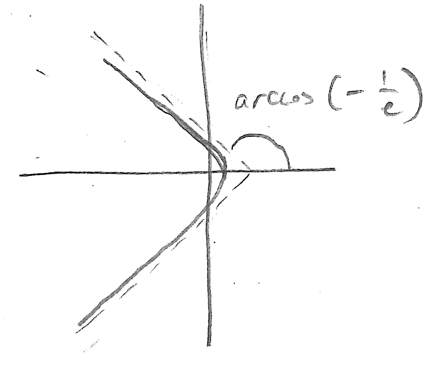
$\phi = \frac{\pi}{2} \Rightarrow r = B$

$\phi = \pi \Rightarrow r = \infty$



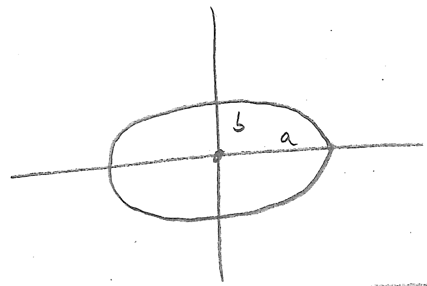
$e > 1$ (hyperbola) $\phi = 0 \Rightarrow r_- = \dfrac{B}{1+e}$

$\phi = \arccos\left(-\frac{1}{e}\right) \Rightarrow r = \infty$



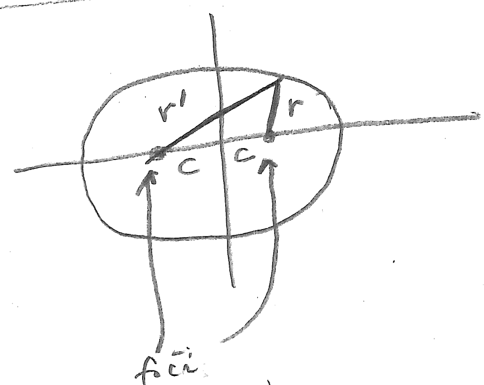
(Kepler's 1st law) planets move in ellipses w/ sun at one focus

Usual Eqn of ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ A5



$a =$ semi major axis
$b =$ semi minor axis

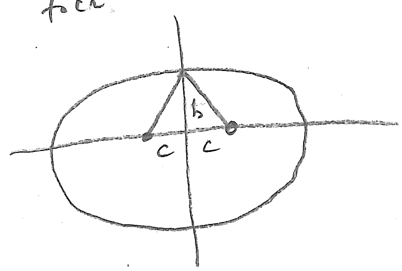
~~pencil~~ pins + string construction



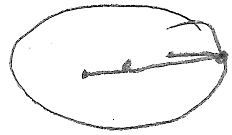
$r + r' = \text{const}$

$c =$ focus

$r + r' = 2\sqrt{b^2 + c^2}$

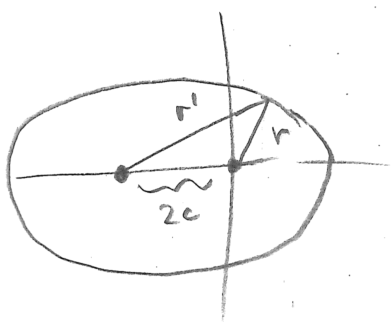


$r + r' = 2a$



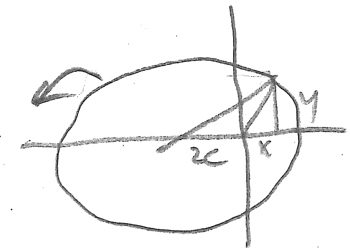
$\Rightarrow \boxed{a^2 = b^2 + c^2}$

Shift origin to one focus



$$r = \sqrt{x^2+y^2}$$

$$r' = \sqrt{(x+2c)^2+y^2}$$



$$r + r' = 2a$$

$$r'^2 = (2a-r)^2$$

$$\cancel{x^2} + 4cx + 4c^2 + \cancel{y^2} = 4a^2 - 4ar + \cancel{r^2}$$

$$ar + cx = a^2 - c^2$$

$$r(a + c\cos\phi) = a^2 - c^2 = b^2$$

$$r = \frac{a^2-c^2}{a + c\cos\phi} = \left(\frac{a^2-c^2}{a}\right)\frac{1}{\left(1 + \underbrace{\frac{c}{a}}_{e}\cos\phi\right)}$$

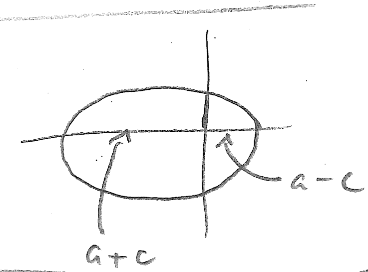
$$c = ea$$

$$\Rightarrow \quad r = \frac{a(1-e^2)}{1+e\cos\phi} \qquad \begin{cases} B = a(1-e^2) \\ e = \frac{c}{a} \end{cases}$$

---

check $\quad r_- = \dfrac{a(1-e^2)}{1+e} = a(1-e) = a - c$

$\qquad r_+ = \dfrac{a(1-e^2)}{1-e} = a(1+e) = a + c$



Also $\quad a = \dfrac{r_+ + r_-}{2} \qquad\qquad e = \dfrac{c}{a} = \dfrac{r_+ - r_-}{r_+ + r_-}$

$\qquad c = \dfrac{r_+ - r_-}{2} \qquad\qquad b = \sqrt{a^2-c^2} = \sqrt{r_+ r_-}$

The ellipse can be characterized in different ways

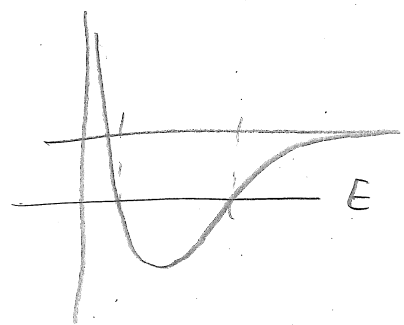
(a, b) $\qquad \dfrac{(x-c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (origin at one focus)

(B, e) $\qquad r = \dfrac{B}{1 + e\cos\phi}$ where $B = \dfrac{\ell^2}{\mu k}$

(r_+, r_-) $\qquad r_\pm = \dfrac{B}{1 \mp e} = \dfrac{a(1-e^2)}{1 \mp e} = a(1 \pm e)$

(E, ℓ) How is E related to the other parameters?

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{U_{eff}(r)}_{-\frac{k}{r} + \frac{\ell^2}{2\mu r^2}}$$



Let $\dot{r} = 0$ to solve for turning points

$$\Rightarrow \quad 0 = E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}$$

$$0 = r^2 + \frac{k}{E} r - \frac{\ell^2}{2\mu E}$$

$$= (r - r_+)(r - r_-)$$

$$= r^2 - [r_+ + r_-]r + r_+ r_-$$

$$\Rightarrow \begin{cases} r_+ + r_- = -\dfrac{k}{E} \\ r_+ r_- = -\dfrac{\ell^2}{2\mu E} \end{cases}$$

$$a = \frac{r_+ + r_-}{2} = -\frac{k}{2E}$$

$$b = \sqrt{r_+ r_-} = \sqrt{-\frac{\ell^2}{2\mu E}}$$

$$c = \sqrt{a^2 - b^2} = a\sqrt{1 - \frac{b^2}{a^2}} = a\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

$$e = \frac{c}{a} = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \Rightarrow \begin{cases} E < 0,\ e < 1 \Rightarrow \text{ellipse} \\ E > 0,\ e > 1 \Rightarrow \text{hyperbola} \end{cases}$$

$$\begin{cases} E = -\frac{k}{2a} = -\frac{k}{(r_+ + r_-)} \\ e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \end{cases}$$

$$r = \frac{B}{1 + e\cos\phi} = \frac{\left(\frac{\ell^2}{\mu k}\right)}{1 + \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\cos\phi}$$

orbit in terms of E and ℓ

Kepler's 2nd law

$$\frac{dA}{dt} = \text{const} = \frac{l}{2\mu}$$

In one period, planet sweeps out area of an ellipse

$$\tau = \frac{2\mu}{l}(\text{area})$$

$$= \frac{2\mu}{l} \pi a b$$

But $b = \sqrt{-\frac{l^2}{2\mu E}} = \sqrt{\frac{l^2 a}{\mu k}}$

$$\Rightarrow \tau = 2\pi\sqrt{\frac{\mu}{k}}\, a^{3/2}$$

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3$$ Kepler's 3rd law

Now $k = G m_1 m_2$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ $\Bigg\} \Rightarrow \tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}$

$\Rightarrow$ ''

depends only on <u>sum</u> of masses

$$r = \frac{\frac{l^2}{\mu k}}{1 + e \cos\phi}$$

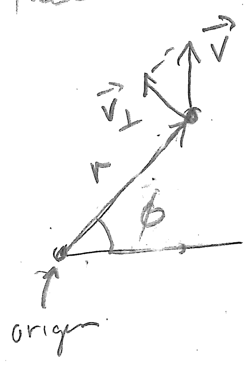
where $\phi = 0$ is perihelion

$$e = \sqrt{1 + \frac{2El^2}{\mu k^2}}$$

$$E = -\frac{k}{2a}$$

$$\tau = 2\pi \sqrt{\frac{\mu}{k}}\, a^{3/2}$$

Orbit parameters are E, l.
These can be determined by initial conditions



$$E = \frac{1}{2}\mu v_0^2 - \frac{k}{r_0}$$

$$l = \mu v_{0\perp} r_0$$

Then ϕ_0 can be determined by

$$r_0 = \frac{\frac{l^2}{\mu k}}{1 + e\cos\phi_0}$$

- v for any point in orbit determined using $E = \frac{1}{2}\mu v^2 - \frac{k}{r}$
- $v_\perp$ for any point in orbit determined using $l = \mu v_\perp r$
  - direction of $\vec{v}$ determined using v + $v_\perp$.

Circular orbit

$$e = 0 \implies E = -\frac{\mu k^2}{2l^2}$$ (E + l are related)

$$E = \frac{1}{2}\mu v_0^2 - \frac{k}{r_0}$$ but $$E = -\frac{k}{2a} = -\frac{k}{2r_0}$$

$$\implies \frac{1}{2}\mu v_0^2 = \frac{k}{2r_0}$$

$$v_0 = \sqrt{\frac{k}{\mu r_0}}$$ (v_0 + r_0 are related)

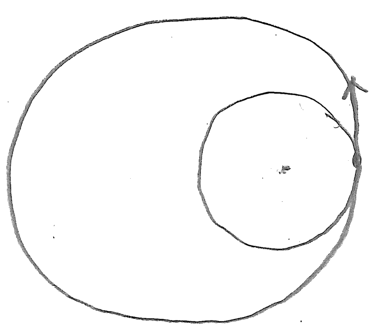
$$\Downarrow$$

$$l = \mu v_0 r_0 = \sqrt{\mu k r_0}$$

Consider a spacecraft in circular orbit about sun
Now boost in various directions

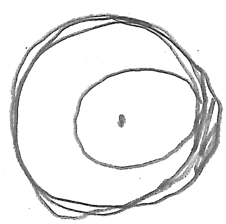
① in direction of motion: V ↑ ⇒ E ↑ ⇒ a ↑
(Also l ↑)

initial position is now perihelion



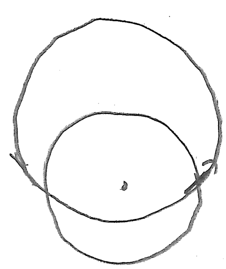
② opposite direction of motion: V ↓ ⇒ E ↓ ⇒ a ↓
(Also l ↓)

initial position is now aphelion



③ in radial direction: V ↑ ⇒ E ↑
but l same

~~initial position is between aphelion & perihelion~~

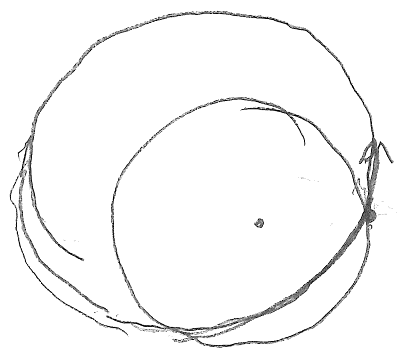


initial orbit $r = \dfrac{\frac{l^2}{\mu k}}{1+0}$

final orbit $r = \dfrac{\left(\frac{l^2}{\mu k}\right)}{1+e\cos\phi}$ (same l)

$\Rightarrow \phi_0 = \frac{\pi}{2}$ ⇒ initial position ⊥ to semi major axis

(4) in any direction



[Ham sandwich]

of the planet. Using the principles of conservation of energy and angular momentum, show that

$$v_0 = (5GM/4R)^{1/2}$$

13–8 A particle moves under the influence of a central *attractive* force, $-k/r^3$. At a very large (effectively infinite) distance away, it has a nonzero velocity that does *not* point toward the center. Construct the effective potential-energy diagram for the radial component of the motion. What conclusions can you draw about the dependence on r of the radial component of velocity?

13–9 A satellite in a circular orbit around the earth fires a small rocket. Without going into detailed calculations, consider how the orbit is changed according to whether the rocket is fired (a) forward; (b) backward; (c) toward the earth; and (d) perpendicular to the plane of the orbit.

13–10 Two spacecraft are coasting in exactly the same circular orbit around the earth, but one is a few hundred yards behind the other. An astronaut in the rear wants to throw a ham sandwich to his partner in the other craft. How can he do it? Qualitatively describe the various possible paths of transfer open to him. (This question was posed by Dr. Lee DuBridge in an after-dinner speech to the American Physical Society on April 27, 1960.)

13–11 The elliptical orbit of an earth satellite has major axis $2a$ and minor axis $2b$. The distance between the earth's center and the other focus is $2c$. The period is T.

(a) Verify that $b = (a^2 - c^2)^{1/2}$.

(b) Consider the satellite at perigee ($r_1 = a - c$) and apogee ($r_2 = a + c$). At these two points its velocity vector and its radius vector are at right angles. Verify that conservation of energy implies that

$$\tfrac{1}{2}mv_1{}^2 - \frac{GMm}{a-c} = \tfrac{1}{2}mv_2{}^2 - \frac{GMm}{a+c} = E$$

Verify also that conservation of angular momentum implies that

$$\frac{\pi ab}{T} = \tfrac{1}{2}(a-c)v_1 = \tfrac{1}{2}(a+c)v_2$$

(c) From the above relationships, deduce the following results, corresponding to Eqs. (13–36) and (13–39) in the text:

$$T^2 = 4\pi^2a^3/GM \qquad \text{and} \qquad E = -GMm/2a$$

13–12 A satellite of mass m is in an elliptical orbit about the earth. When the satellite is at its perigee, a distance R_0 from the center of