Euler Lagrange eqn for radial coordinate

$$\frac{\partial L}{\partial r} = \mu r \dot{\phi}^2 - \frac{dU}{dr} \qquad \text{but} \quad \dot{\phi} = \frac{l}{\mu r^2}$$

$$= \frac{l^2}{\mu r^3} - \frac{dU}{dr}$$

$$\frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$

really

$$\mu a_r = \mu(\ddot{r} - r^2\dot{\phi}^2) = -\frac{dU}{dr}$$

$$\Rightarrow \quad \mu \ddot{r} = \underbrace{\frac{l^2}{\mu r^3}}_{\text{"centrifugal force"}} - \underbrace{\frac{dU}{dr}}_{\text{physical force}}$$

"centrifugal force" (because we are treating a 2d problem as a 1d radial problem)

$$\mu \ddot{r} = -\frac{d}{dr}\Big(\underbrace{\frac{l^2}{2\mu r^2} + U}_{U^{eff}}\Big) = F_r^{eff}$$

Multiply by $\dot{r}$

$$\mu \dot{r} \ddot{r} = -\frac{dU^{eff}}{dr}\frac{dr}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right) = -\frac{d}{dt}U^{eff}$$

$$\boxed{\frac{1}{2}\mu \dot{r}^2 + U^{eff} = E}$$

use 1d potential energy diagrams

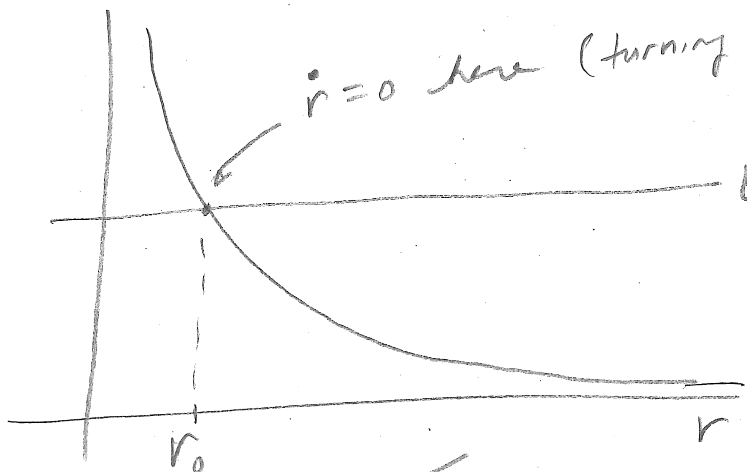
where $U^{eff} = U + \frac{l^2}{2\mu r^2}$

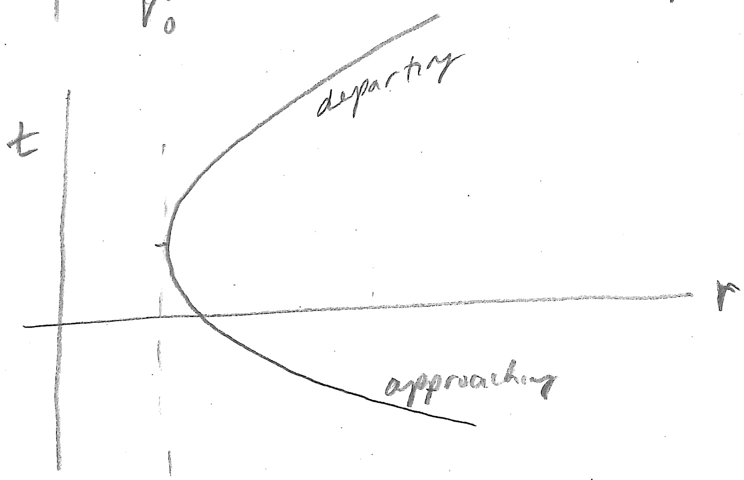
this is "really" kinetic energy in azimuthal (ϕ) direction

First suppose there is no physical force ($U = 0$)

$$U^{eff} = \frac{\ell^2}{2\mu r^2}$$

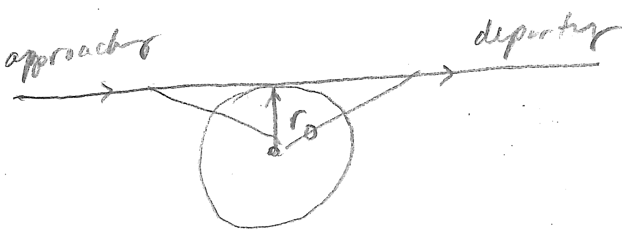
$\dot{r} = 0$ here (turning point) → but not <u>at rest</u> because there is motion in azimuthal direction!





From a 1d viewpoint there is a repulsive radial force

From a 2d viewpoint, no force



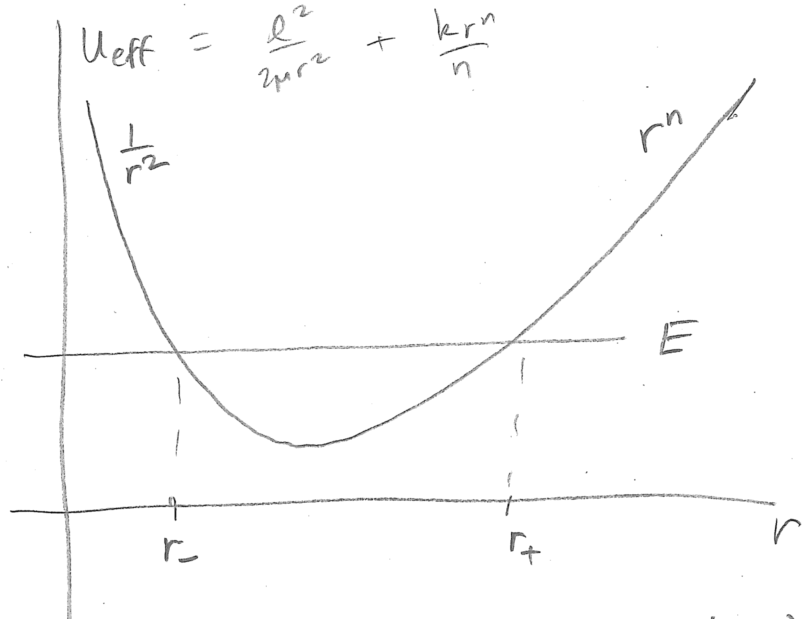
Suppose there is an attractive power law force

$$F_r = -kr^{n-1} \quad \Rightarrow \quad U = \frac{kr^n}{n} \qquad \left(F_r = -\frac{dU}{dr}\right)$$

e.g.

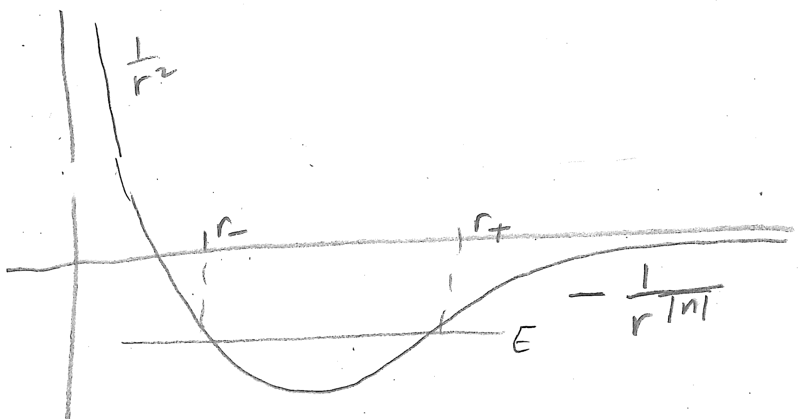
n	F_r	U	
2	$-kr$	$\frac{1}{2}kr^2$	(Hooke's law)
1	$-k$	kr	(constant) strong force for quarks
0	$-\frac{k}{r}$	$k \log r$	
-1	$-\frac{k}{r^2}$	$-\frac{k}{r}$	(Inverse square)

For $n > 0$



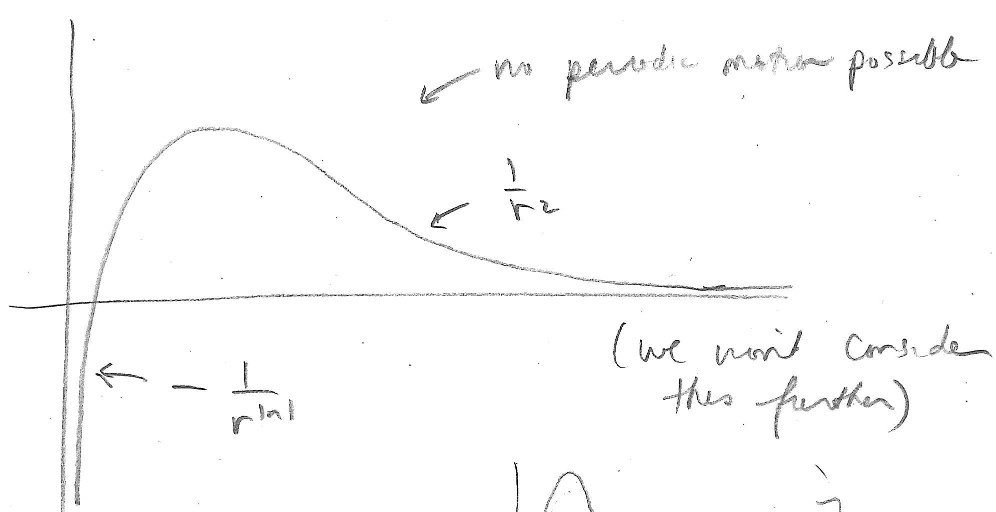
(periodic motion between r_- and r_+)

For $-2 < n < 0$

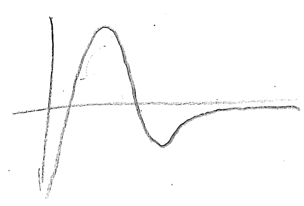


periodic motion if $E < 0$
(unbound motion if $E > 0$)

For $n < -2$

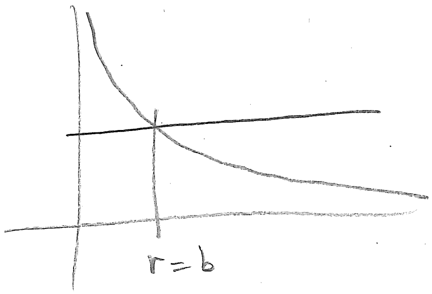


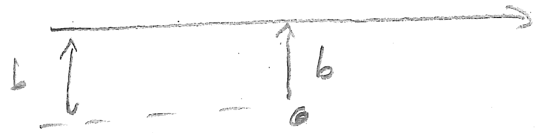
(we won't consider this further)

[Black hole ]

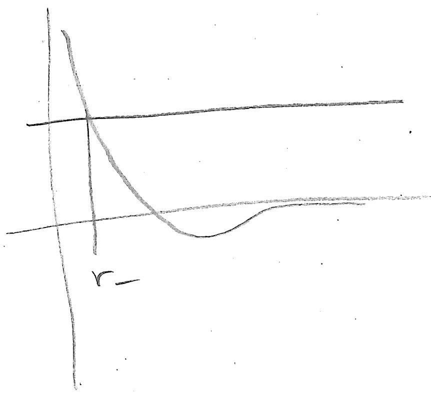
Let $n<0$

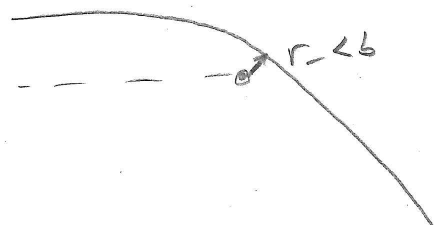
$k=0$
no force

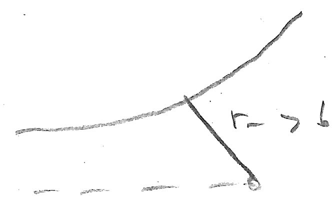




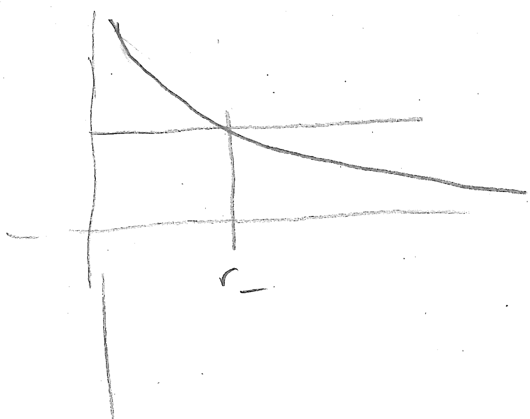
$k>0$
attraction







$k<0$
repulsion



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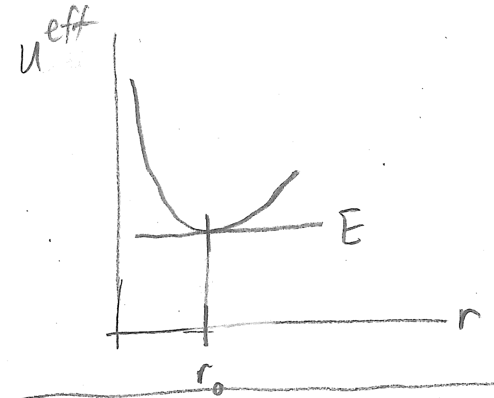
Let $n > -2$.

$$U^{eff} = \frac{\ell^2}{2\mu r^2} + \frac{k r^n}{n}$$ has a minimum for some value r_0

$$\frac{dU^{eff}}{dr} = -\frac{\ell^2}{\mu r^3} + k r^{n-1} = 0 \quad \Rightarrow \quad r_0^{\,n+2} = \frac{\ell^2}{k\mu}$$



Suppose $E = U^{eff}_{min}$

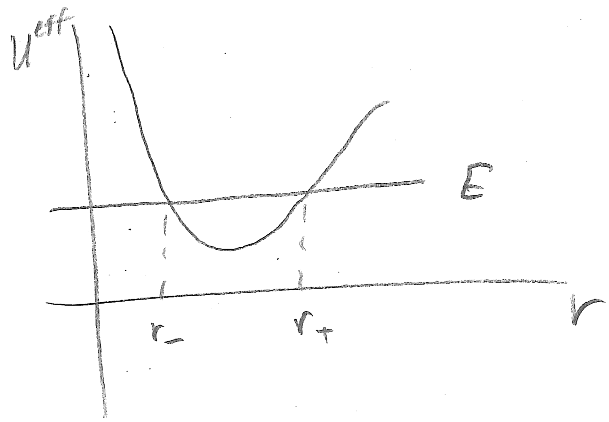
Then $\dot{r} = 0 \Rightarrow$ circular orbit

$$\dot{\phi} = \frac{\ell}{\mu r_0^2} = \text{const}$$

$$\phi = \frac{\ell}{\mu r_0^2} t + \phi_0$$

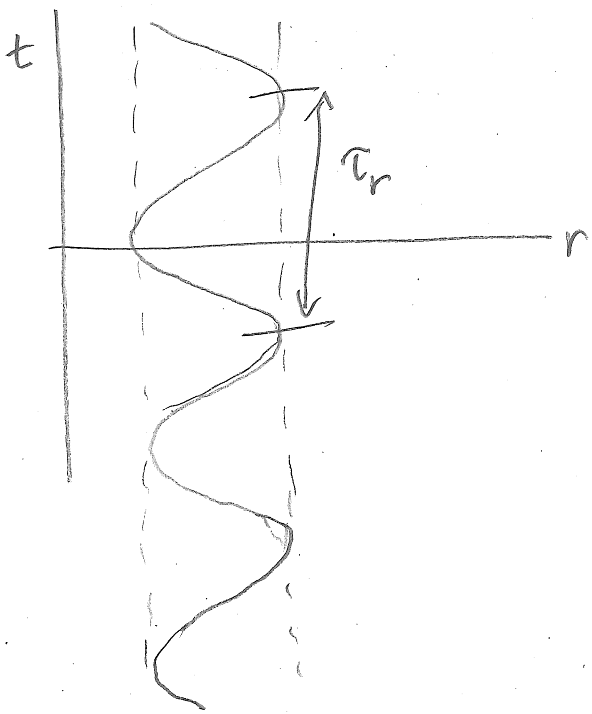
$\Rightarrow$ orbital period $\tau_\phi = \dfrac{2\pi \mu r_0^2}{\ell}$

Next, suppose $E > U^{eff}_{min}$



Periodic radial motion between r_- and r_+

w/ radial period τ_r



$$\frac{1}{2}\mu\dot{r}^2 + U^{eff}(r) = E$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U^{eff})}$$

$$\int dt = \int \frac{dr}{\sqrt{\frac{2}{\mu}(E - U^{eff})}}$$

$$t = \int^r \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} + \frac{kr^n}{n}\right)}}$$

Solve numerically or analytically.
Invert to get $r(t)$.

$$\tau_r = 2\int_{r_-}^{r_+} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} + \frac{kr^n}{n}\right)}}$$

(depends on E and ℓ)

Orbital period T_ϕ

$$\frac{d\phi}{dt} = \frac{l}{\mu r^2}$$

If we have an explicit form for $r(t)$

$$\int d\phi = \int \frac{l\,dt}{\mu r^2(t)}$$

$$\phi(t) - \phi_0 = \int_0^t \frac{l\,dt}{\mu r^2(t)}$$

$$\phi(T_\phi) = \phi_0 + 2\pi \quad \text{so}$$

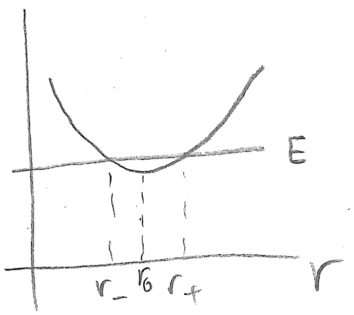
$$2\pi = \int_0^{T_\phi} \frac{l\,dt}{\mu r^2(t)}$$

Solve this eqn to obtain T_ϕ.

If $r(t)$ remains <u>close</u> to r_0 then we have approximately

$$2\pi \approx \int_0^{T_\phi} \frac{l}{\mu r_0^2}\,dt \quad \Rightarrow \quad T_\phi \approx \frac{2\pi\mu r_0^2}{l}$$

Suppose E is slightly larger than U^{eff}_{min}.



We can approximate U^{eff} as a parabola near the minimum

$$U^{eff} \approx \frac{1}{2}\lambda\,(r-r_0)^2$$

The radial motion is simple harmonic w/ $\omega = \sqrt{\frac{\lambda}{\mu}}$

and period $\quad T_r \approx 2\pi\sqrt{\frac{\mu}{\lambda}}$

λ = curvature of U^{eff} at r_0

$$= \left.\frac{d^2U^{eff}}{dr^2}\right|_{r_0} = \frac{3\ell^2}{\mu r_0^4} + k(n-1)\,r_0^{\,n-2}$$

$$= \frac{3\ell^2}{\mu r_0^4}\left[1 + \frac{\mu k(n-1)}{3\ell^2}\,r_0^{\,n+2}\right]$$

But $r_0^{\,n+2} = \frac{\ell^2}{\mu k}$ from earlier

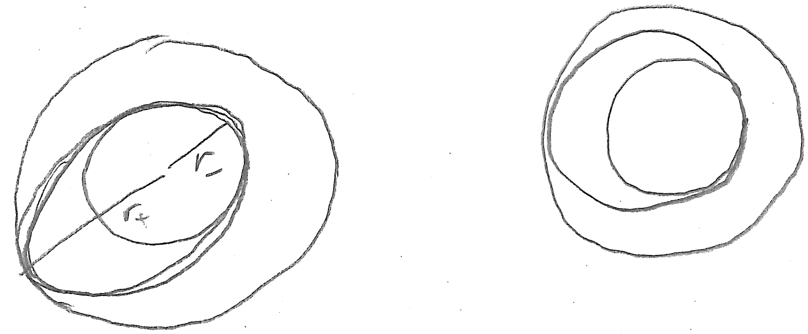
$$\lambda = \frac{3\ell^2}{\mu r_0^4}\left[1 + \frac{n-1}{3}\right] = \frac{(n+2)\,\ell^2}{\mu r_0^4}$$

$$T_r \approx 2\pi\,\frac{\mu r_0^2}{\ell\sqrt{n+2}}$$

Approximate orbital and radial periods
(for motion near the minimum of U_{eff})

$$T_\phi \approx \frac{2\pi \mu r_0^2}{l} \qquad T_r \approx \frac{2\pi \mu r_0^2}{l\sqrt{n+2}}, \quad \Rightarrow \frac{T_r}{T_\phi} = \frac{1}{\sqrt{n+2}}$$

For inverse square law $(n=-1)$, periods are equal.
$T_r \approx T_\phi$
Orbit closes!



Although we've only shown this for
slight deviations from circular, it remains
true for exact solution $\Rightarrow$ (ellipses w/ center at focus)

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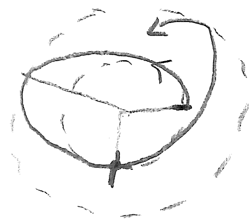
For $n>-1$, $\frac{T_r}{T_\phi} < 1$



orbit does not close unless $\sqrt{n+2}$ is rational (ie integer)

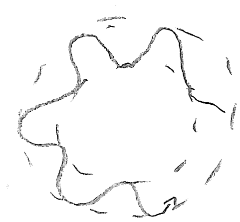
$n=2 \Rightarrow \frac{T_r}{T_\phi} = \frac{1}{2}$

Again, although we've only shown it approximately this result is exact

$n=2 \Rightarrow$ 2d harmonic oscillator $U=\frac{1}{2}k r^2 = \frac{1}{2}k(x^2+y^2)$

= two 1d oscillators w/ <u>same</u> frequency!

For large n, $T_r \ll T_\phi$



For $-2<n<-1$, $\frac{T_r}{T_\phi} > 1$

Bertrand's thm: only central forces that result in closed orbits are $n=-1$ and $n=2$