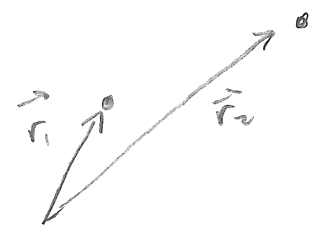
Two-body central force problem in 3 dimensions

6 d.o.f. $\vec{r}_1$ and $\vec{r}_2$



$$L = T - U$$

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

Recall $\quad = \tfrac{1}{2} M V_{cm}^2 + \tfrac{1}{2} \mu v^2$

where
$$\begin{cases} M = m_1 + m_2 \\ \dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2} \\ \vec{V}_{cm} = \dfrac{m_1}{M}\vec{v}_1 + \dfrac{m_2}{M}\vec{v}_2 \\ \vec{v} = \vec{v}_1 - \vec{v}_2 \end{cases}$$

$$\vec{r}_{cm} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

No external forces

Internal force $\vec{F}_{12}$ depends on $\vec{r}_{12} = \vec{r}$

Conservative force $\vec{F}_{12} = -\vec{\nabla}_1 U(\vec{r}_{12}) = \vec{\nabla}_2 U(\vec{r}_{12}) = -\vec{F}_{21}$

$$L = \tfrac{1}{2} M \dot{\vec{r}}_{cm}^{\,2} + \tfrac{1}{2}\mu \dot{\vec{r}}^{\,2} - U(\vec{r})$$

we can use $\vec{r}_{cm}$ and $\vec{r}$
as generalized coordinates

L is independent of $\vec{r}_{cm}$

$\Rightarrow$ conjugate momentum $\dfrac{\partial L}{\partial \dot{\vec{r}}_{cm}} = M\dot{\vec{r}}_{cm} = \vec{P}_{cm}$ is conserved.

$\Rightarrow \quad \vec{r}_{cm}(t) = \vec{r}_{cm}(0) + \vec{V}_{cm}\, t$

We have reduced two-body problem to a one-body problem

$$L = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(\vec{r})$$

"Central force" $\Rightarrow$ $\vec{F}_{12}$ parallel to $\vec{r}$

$\Rightarrow$ $U(\vec{r})$ only depends on $r = |\vec{r}|$

(otherwise $\vec{\nabla} U = \left(\dfrac{\partial U}{\partial r}, \dfrac{1}{r}\dfrac{\partial U}{\partial \theta}, \dfrac{1}{r\sin\theta}\dfrac{\partial U}{\partial \phi}\right)$ would have non radial components)

$$L = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(r)$$

Angular momentum review

$$\vec{L}^{sys} = \vec{L}_1 + \vec{L}_2$$

$$= \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{L}'^{\,sys}$$

where

$$\vec{L}'^{\,sys} = \vec{r} \times \mu\vec{v}$$ = ang. mom. evaluated in cm frame (and also ang. mom. of equivalent 1-body problem)

We proved (assuming all internal forces are central)

$$\frac{d\vec{L}'^{\,sys}}{dt} = \vec{\tau}'^{\,ext}$$

but there are no external forces (or torques) in our two body problem so

$\vec{L}'^{\,sys}$ is conserved

We can check this explicitly:

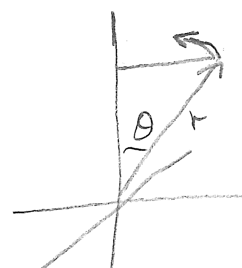
$$\frac{d\vec{L}'^{\,sys}}{dt} = \underbrace{\vec{v} \times \mu\vec{v}}_{0} + \vec{r} \times \underbrace{\mu\dot{\vec{v}}}_{\vec{F}_{12}} = 0$$ because $\vec{F}_{12} \parallel \vec{r}$

In spherical coordinates

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r)$$

L is independent of ϕ

$\Rightarrow$ conjugate mom $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \sin^2\theta\,\dot{\phi}$ = conserved



But $L_z = \mu r_\perp^2 \dot{\phi} = \mu r^2 \sin^2\theta\,\dot{\phi}$

so L_z is conserved.

But choice of z-axis was arbitrary,

so angular mom about any axis is conserved

$\Rightarrow \vec{L}$ = conserved

In particular, the direction of $\vec{L}$ is conserved

Since $\vec{L} = \vec{r} \times \mu\vec{v}$, the direction is $\perp$ to the plane defined by $\vec{r}$ and $\vec{v}$

NB $\Delta\vec{v}$ is $\parallel$ $\vec{r}$ & so in the same plane

$\therefore$ the motion remains confined to the plane

central-force motion is planar

Choose the plane of motion to be the xy-plane
(ie choose the z-axis to be parallel to the
[conserved] angular momentum)

$\therefore\ \theta = \frac{\pi}{2}$ in spherical coords.

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

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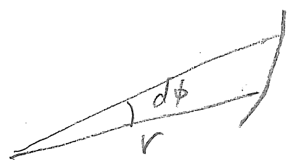
angular momentum (= conjugate to ϕ) $\ell = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{const}$ because $\frac{\partial L}{\partial \phi} = 0$

Angular momentum conservation is equivalent to Kepler's 2nd law of planetary motion:
<u>equal area</u> swept out in equal time



$$dA = \frac{1}{2} r (r\, d\phi)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{\ell}{2\mu}$$

true for any central force (not just $1/r^2$)

~~Time for one orbital period~~

$$\cancel{\tau_\phi = dt = \frac{d\phi}{\dot{\phi}} = \frac{d\phi\, \mu r^2(\phi)}{p_\phi}}$$

$$\cancel{\frac{\mu}{p_\phi}\int_0^{2\pi} r^2(\phi)\, d\phi}$$

[xked]

(9-25-17)

Routhian

Central force in 2d

$$L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} = p \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\text{E-L} \Rightarrow \quad mr^2\dot{\theta} = l = \text{const}$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$= \frac{l^2}{mr^3} - \frac{\partial V}{\partial r} \checkmark$$

naively

$$L = \tfrac{1}{2}m\dot{r}^2 + \overbrace{\frac{l^2}{2mr^2} - V(r)}^{\text{Note this is not } -V_{eff}!}$$

$$\frac{\partial L}{\partial r} = -\frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$m\ddot{r} = -\frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

wrong sign!!

Also $V_{eff} = V + \frac{l^2}{2mr^2}$ gives then

$$H = p\dot{r} + l\dot{\theta} - L$$

$$= \frac{p^2}{2m} + \frac{l^2}{2mr^2} + V(r)$$

$$\dot{\theta} = \frac{\partial H}{\partial l} = \frac{l}{mr^2}$$

$$\dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{l} = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot{p} = -\frac{\partial H}{\partial r} = +\frac{l^2}{mr^3} - \frac{\partial V}{\partial r} \checkmark$$

The correct way to implement this is via the Routhian

$$R = L - l\dot{\theta}$$

$$= \tfrac{1}{2}m\dot{r}^2 \overbrace{- \frac{l^2}{2mr^2} - V(r)}^{\text{This } \underline{\underline{\text{is}}}\ V_{eff}(r)} \quad \text{so } R = T - V_{eff}$$

Then w.r.t. θ we use Hamilton's eq's

$$\dot{\theta} = -\frac{\partial R}{\partial l} = +\frac{l}{mr^2}, \quad \dot{l} = +\frac{\partial R}{\partial \theta} = 0$$

(because I defined R as negative of the usual)

and w.r.t. r we use Euler-Lagrange

$$\frac{dR}{d\dot{r}} = m\dot{r}$$

$$\frac{dR}{dr} = +\frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\Rightarrow m\ddot{r} = \frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

correct sign!

I finally get the point of the Routhian!