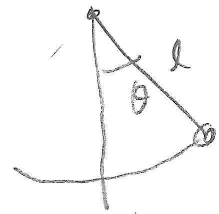
Constrained systems

① Simple plane pendulum



(particle constrained to move on a circle)
1 d.o.f

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r, \theta)$$

constraint $r = l \Rightarrow \dot{r} = 0$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

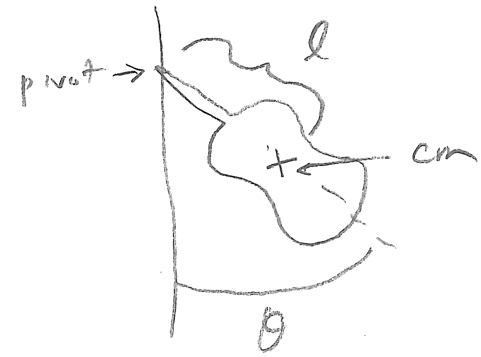
$$\frac{\partial L}{\partial \theta} = - m g l \sin\theta$$

Euler-Lagrange $\Rightarrow$ $m l^2 \ddot{\theta} + m g l \sin\theta = 0$

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$ as we found earlier

(2) <u>physical pendulum</u>

1 d.o.f.



$$L = T - U$$

$$= \frac{1}{2} I \dot{\theta}^2 + mgl\cos\theta$$

I = mom. of inertia about pivot point

Euler-Lagrange: $\quad I\ddot{\theta} + mgl\sin\theta = 0 \qquad$ as before

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Observe: $\quad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} = -L_y \quad$ ← angular momentum

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta = -\tau_y \quad \leftarrow \text{torque}$$

generalized momentum

generalized force

Thus Euler-Lagrange $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$

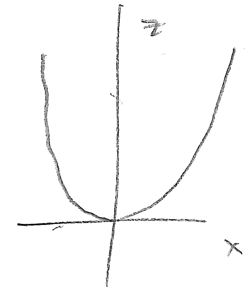
$$\Rightarrow \quad \frac{dL_y}{dt} = \tau_y$$

Recall corollary: if L indep of θ then $\frac{\partial L}{\partial \dot{\theta}}$ = const

L indep of θ ⇒ no torque ⇒ angular moment is conserved

[Problem: double pendulum]

③ Bead constrained to move on a parabolic wire
(1 d.o.f.)



Constraint: $z = cx^2$

$L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$

$\dot{z} = \frac{dz}{dx}\dot{x} = 2cx\dot{x}$

$L = \frac{1}{2}m(1+4c^2x^2)\dot{x}^2 - mgcx^2$

$\frac{\partial L}{\partial \dot{x}} = m(1+4c^2x^2)\dot{x}$, $\quad \frac{\partial L}{\partial x} = 4mc^2x\dot{x}^2 - 2mgcx$

E-L $\Rightarrow$ $\underbrace{\frac{d}{dt}[m(1+4c^2x^2)\dot{x}]} - 4mc^2x\dot{x}^2 + 2mgcx = 0$

$m(1+4c^2x^2)\ddot{x} + 8mc^2x\dot{x}^2$.

$$\boxed{m(1+4c^2x^2)\ddot{x} + 4mc^2x\dot{x}^2 + 2mgcx = 0}$$

Can we integrate this? Yes. Multiply by $\dot{x}$

$$m(1+4c^2x^2)\dot{x}\ddot{x} + 4mc^2x\dot{x}^3 + 2mgcx\dot{x} = 0$$

$$= \frac{d}{dt}\left[\frac{1}{2}m(1+4c^2x^2)\dot{x}^2 + mgcx^2\right] = 0$$

How did we know this was possible?

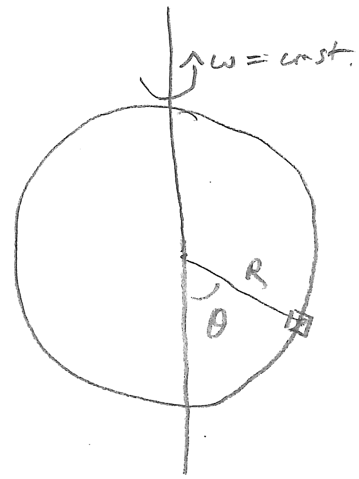
L independent of t so 2nd form of Euler's eqn gives energy conserved

$$\dot{x} = \sqrt{\frac{2(E - mgcx^2)}{m(1+4c^2x^2)}}$$

[NB. not quite a harmonic oscillator]

$$\int dx \sqrt{\frac{m(1+4c^2x^2)}{2(E - mgcx^2)}} = \int dt = t$$

(4) Bead on a rotating hoop (1 d.o.f.)
[Taylor, p. 260]



$z = -R\cos\theta$

$x = R\sin\theta\cos\omega t$

$y = R\sin\theta\sin\omega t$

$\dot z = +R\sin\theta\,\dot\theta$

$\dot x = R\cos\theta\,\dot\theta\cos\omega t - \omega R\sin\theta\sin\omega t$

$\dot y = R\cos\theta\,\dot\theta\sin\omega t + \omega R\sin\theta\cos\omega t$

$\dot x^2 + \dot y^2 + \dot z^2 = R^2\cos^2\theta\,\dot\theta^2 + \omega^2R^2\sin^2\theta + R^2\sin^2\theta\,\dot\theta^2$

$= R^2\dot\theta^2 + R^2\omega^2\sin^2\theta$

$L = T - U = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta$

Euler-Lagrange: $mR^2\ddot\theta - mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta = 0$

$$\ddot\theta = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)$$

Look for equilibrium position $\theta = \theta_0 \Rightarrow \ddot\theta = 0$

Either $\sin\theta_0 = 0 \Rightarrow \theta = 0$ or π

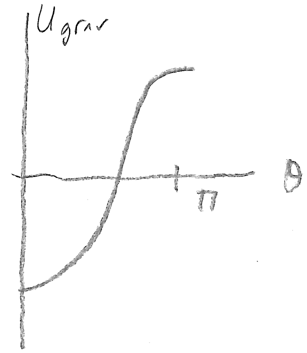
or $\cos\theta_0 = \frac{g}{\omega^2R} \Rightarrow$ possible only if $\omega^2 > \frac{g}{R}$

Stable or unstable?

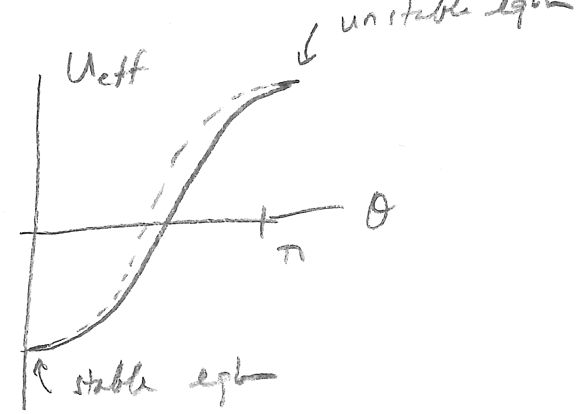
Since L doesn't depend explicitly on t,

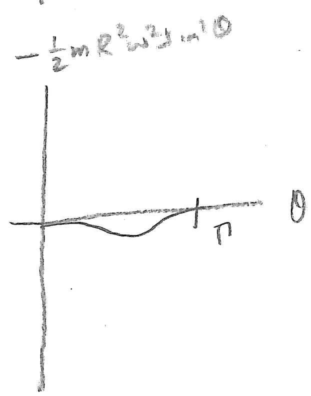
$$\underbrace{\dot\theta \frac{\partial L}{\partial \dot\theta}}_{mR^2\dot\theta^2} - L = \text{const} = E$$

$$E = \frac{1}{2} mR^2 \dot\theta^2 \underbrace{\underbrace{- mgR\cos\theta}_{U_{grav}(\theta)} - \frac{1}{2} mR^2\omega^2 \sin^2\theta}_{U_{eff}(\theta)}$$

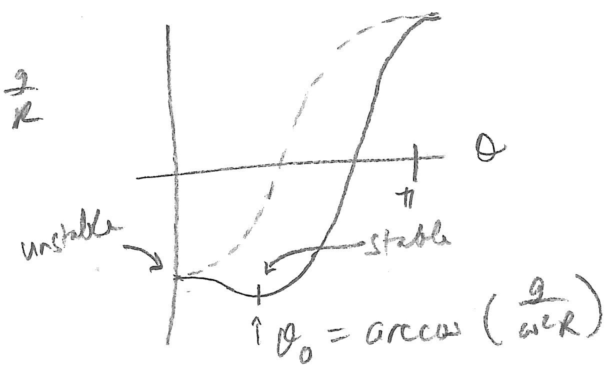


If $\omega^2 < \frac{g}{R}$



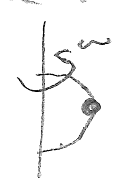


If $\omega^2 > \frac{g}{R}$



[Imagine starting with ω small & gradually increasing]

As $\omega \uparrow$, bead moves up the wire

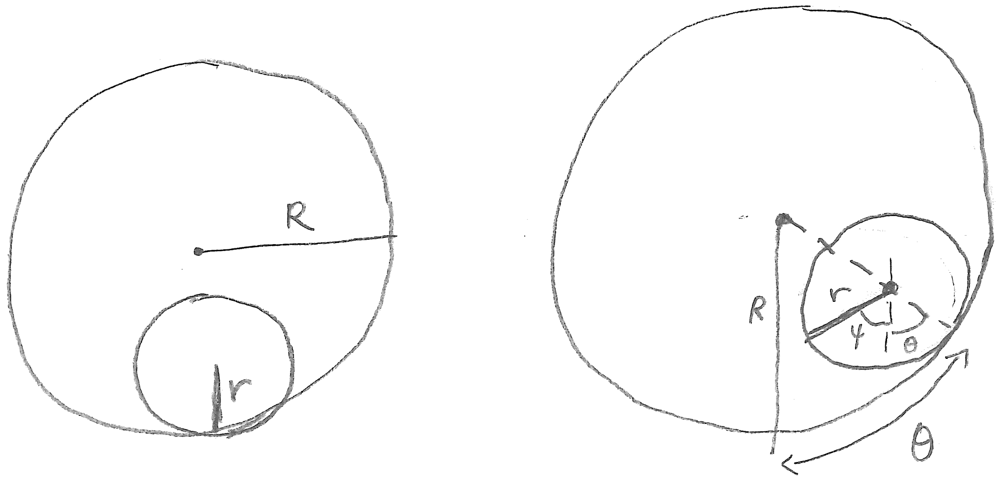


As $\omega \to \infty$, $\theta_0 \to \frac{\pi}{2}$

James Watt (1736-1819) used such a device as a regulator to control speed of steam engine

[cf Taylor p. 264]

(5) Pipe rolling inside a larger pipe (w/o slipping)



ψ = angle inner pipe has turned wrt vertical.

Note $R\theta = r(\psi + \theta) \quad \Rightarrow \quad \psi = \left(\frac{R-r}{r}\right)\theta$

$$T = T_{cm} + T'$$

$$= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I' \dot{\psi}^2$$

mom. of inertia about c.m.

$$v_{cm} = (R-r)\dot{\theta}$$

$$I' = mr^2 \quad \Rightarrow \quad \frac{1}{2} I' \dot{\psi}^2 = \frac{1}{2} m (R-r)^2 \dot{\theta}^2$$

$$T = m(R-r)^2 \dot{\theta}^2$$

$$U = -mg(R-r)\cos\theta$$

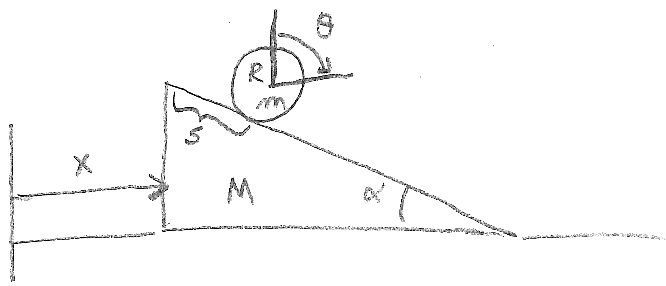
Euler Lagrange:
$$2m(R-r)^2 \ddot{\theta} + mg(R-r)\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{2(R-r)}\sin\theta = 0$$

or energy approach

[cf Taylor, 259]
[Marion, 3e, Prob 6.6]

⑥ Round object rolling (w/o slipping) down an inclined plane resting on a frictionless surface (2 d.o.f.)



$\theta = 0$ at top of incline

center of mass of round object

$$\begin{cases} x_{cm} = x + s\cos\alpha \\ z_{cm} = z_0 - s\sin\alpha \end{cases}$$

$$s = R\theta$$

$$\Rightarrow \begin{cases} \dot{x}_{cm} = \dot{x} + R\dot{\theta}\cos\alpha \\ \dot{z}_{cm} = -R\dot{\theta}\sin\alpha \end{cases}$$

$$\Rightarrow v_{cm}^2 = \dot{x}^2 + R^2\dot{\theta}^2 + 2R\cos\alpha\,\dot{x}\dot{\theta}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\dot{\theta}^2 - mg z_{cm}$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(mR^2 + I)\dot{\theta}^2 + mR\cos\alpha\,\dot{x}\dot{\theta} + (mgR\sin\alpha)\,\theta - mgz_0$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + mR\cos\alpha\,\dot{\theta}$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{x}} = \text{const}$$

Observe that $\frac{\partial L}{\partial \dot{x}} = p_x^{sys}$

which is conserved because $F_x^{ext} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = (mR^2 + I)\dot{\theta} + mR\cos\alpha\,\dot{x}$$

$$\frac{\partial L}{\partial \theta} = mgR\sin\alpha$$

$$\text{E-L} \Rightarrow \quad (mR^2 + I)\ddot{\theta} + mR\cos\alpha\,\ddot{x} - mgR\sin\alpha = 0$$

Since $p_x = \text{const} \Rightarrow \quad \ddot{x} = -\frac{mR\cos\alpha}{M+m}\ddot{\theta}$

$$\left[(mR^2 + I) - \frac{(mR\cos\alpha)^2}{M+m}\right]\ddot{\theta} = mgR\sin\alpha$$

Since $s = R\theta$, we can write

$$\ddot{s} = \frac{g\sin\alpha}{\left[1 + \frac{I}{mR^2} - \frac{m\cos^2\alpha}{M+m}\right]}$$

[Consider various limits]