## Lagrangian approach to mechanics.

1788
Lagrange (1736-1813)

- an elegant alternative to Newtonian approach
  based on a minimum principle,
  the principle of least action

- easier because uses scalar rather than vector

- especially useful for systems involving constraints
  or in non Cartesian coordinates

- essential for modern understanding of QM + QFT

Define a scalar function called Lagrangian

$$L = T - U$$ [not energy!]

for a non relativistic, conservative system [Goldstein (1e) 206]

[can be generalized to relativistic, non cons.]

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Let $x_i$ be Cartesian coordinates of a point particle in a potential.

$$L = \frac{1}{2} m \sum_{j=1}^{3} \dot{x}_j^2 - U(x_i)$$

$$\begin{cases} x_1 = x \\ x_2 = y \\ x_3 = z \end{cases}$$

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Consider partial derivatives

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i, \qquad \frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} = F_i$$

Newton 2nd law can be rewritten

$$m\ddot{x}_i = F_i \qquad i = 1, 2, 3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

But these are just Euler eqns for a $\underbrace{\text{path in time}}$ $x_i(t)$ that extremizes $\int L \, dt$

call this a "motion"

Define the *action* as

$$S[x_i(t)] = \int_{t_0}^{t_1} dt\, L(x_i, \dot{x}_i)$$

*Principle of least action* (Hamilton 1834)

the classical motion of a point particle $x_i(t)$ between fixed endpoints $x_i(t_0)$ and $x_i(t_1)$ minimize the action

$\Rightarrow$ Euler-Lagrange eqns: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$

$\Downarrow$

Newton's 2nd law.

[actually, principle of stationary action]

A reformulation of Newton's laws that is independent of the coordinate system!

Since $L$ is a scalar, we can rewrite it in coordinates other than Cartesian, or indeed in terms of any set of <u>generalized coordinates</u> <u>$q_i$</u> equal to the number of degrees of freedom of the system.

$$S = \int_{t_0}^{t_1} dt\, L(q_i, \dot{q}_i)$$

$\dot{q}_i$ = generalized velocities

Space of generalized coordinates is called <u>configuration space</u>.

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Principle of stationary action for generalized coordinates
$\Rightarrow$ Euler-Lagrange eqns

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

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Define generalized momenta $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ (momentum conjugate to $q_i$)

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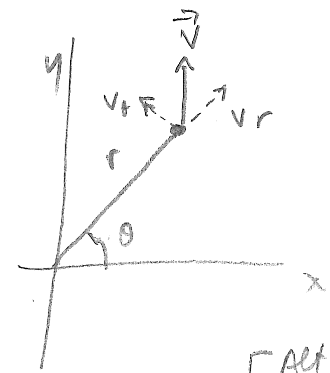
Corollary: if $L$ is independent of one (or more) of the generalized coordinates $\left(\frac{\partial L}{\partial q_i} = 0\right)$ the corresponding generalized momentum is conserved $\left(\frac{dp_i}{dt} = 0\right)$

$\longrightarrow$ $\frac{\partial L}{\partial q_i}$ = generalized force $F_i$ $\Rightarrow$ $F_i = \frac{dp_i}{dt}$

# Motion of a particle in a plane using polar coordinates

generalized coordinates are $r, \theta$




$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$
$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

[Alt, can use
$x = r\cos\theta$
$y = r\sin\theta$
+ chain rule]

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r,\theta).$$

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Characterize motion as $r(t)$, $\theta(t)$

Varying these functions, action is minimized

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0, \qquad \frac{d}{dt}(mr^2\dot{\theta}) + \frac{\partial U}{\partial \theta} = 0$$

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r}, \qquad m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = -\frac{\partial U}{\partial \theta}$$

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Now $\quad \vec{F} = -\vec{\nabla} U = \left(-\frac{\partial U}{\partial r}, -\frac{1}{r}\frac{\partial U}{\partial \theta}\right) = (F_r, F_\theta)$

So Euler eqs are

$$m(\ddot{r} - r\dot{\theta}^2) = F_r \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$$

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These are just $\vec{F} = m\vec{a}$

where $\Rightarrow \quad a_r = \ddot{r} - \underbrace{r\dot{\theta}^2}_{\text{centripetal}} \qquad a_\theta = r\ddot{\theta} + \underbrace{2\dot{r}\dot{\theta}}_{\text{coriolis}}$

easy way to derive acceleration in non Cartesian coordinates

What about 2nd form of Euler's eqn?

$$\frac{d}{dp}\left( x' \frac{\partial f}{\partial x'} - f \right) + \frac{\partial f}{\partial p} = 0$$

Here $p \to t$

$f \to L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r,\theta)$

$x' \frac{\partial f}{\partial x'} \to \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} = m\dot{r}^2 + m r^2 \dot{\theta}^2 = 2T$

$$\Rightarrow \frac{d}{dt}\Big( \underbrace{2T - [T - U]}_{\underbrace{T+U}_{E_{mech}}} \Big) + \frac{\partial L}{\partial t} = 0$$

$$\frac{dE_{mech}}{dt} + \frac{\partial L}{\partial t} = 0$$

If $L$ does not depend explicitly on $t$ then mechanical energy is conserved

For any $T = f(q)\dot{q}^2$

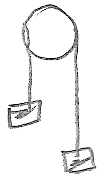
$x' \frac{\partial L}{\partial x'} = 2T$

so we get $2T - (T - V) = T + V$

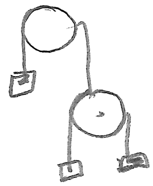
NB. There is only one 2nd form of Euler but several 1st form of Euler (one for every variable)

Atwood

Goldstein
Taylor

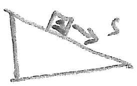


Marion

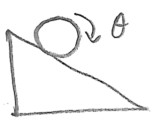


Slide down plane

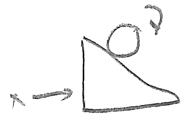
Eck




Roll down plane (fixed)
[Taylor 7-16]




frictionless plane
(Taylor w/ block, ex. 7.5)

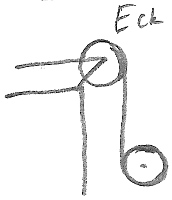



problem

in class

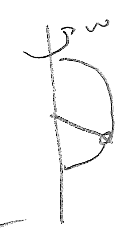
Unwinding spool

Eck

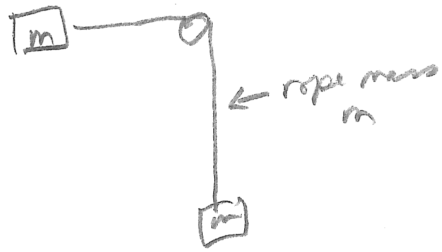


Spinning hoop

(Taylor)




in class




M+T 6-10


M+T 6-12