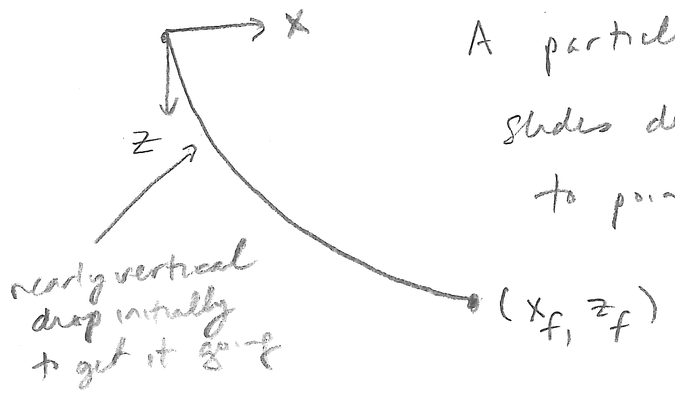
Brachistochrone (shortest time)

Johann Bernoulli posed this in 1696 as a challenge
- Newton solved it in a day
- Leibniz, l'Hôpital
Bernoullis also solved



A particle starting at rest at $(0,0)$ slides down a frictionless wire to point (x_f, z_f). What shape minimizes the time it takes?

$$\frac{ds}{dt} = v$$ where ds = arclength along wire

$$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dz^2}}{v}$$

Energy conservation $\Rightarrow E = \frac{1}{2}mv^2 - mgz = \text{const}$

initially $v = 0$ and $z = 0$ $\Rightarrow E = 0$

$$\Rightarrow v = \sqrt{2gz}$$

$$T = \int dt = \int \sqrt{\frac{dx^2 + dz^2}{2gz}}$$

can ignore $\frac{1}{\sqrt{2g}}$

Can choose either x or z as independent variable p

$$T = \int \sqrt{\frac{1+\left(\frac{dz}{dx}\right)^2}{z}}\, dx \quad \text{or} \quad \int \sqrt{\frac{\left(\frac{dx}{dz}\right)^2+1}{z}}\, dz$$

In this case $f(z, \frac{dz}{dx}, x)$ is independent of x so use 2nd form corollary

$$\Rightarrow z' \frac{\partial f}{\partial z'} - f = \text{const}$$

In this case $f(x, \frac{dx}{dz}, z)$ is indep of x so use 1st form corollary

$$\frac{\partial f}{\partial x'} = \text{const}$$

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{z(1+z'^2)}}$$

$$\frac{z'^2}{\sqrt{z(1+z'^2)}} - \sqrt{\frac{1+z'^2}{z}} = c'$$

$$\frac{z'^2 - (1+z'^2)}{\sqrt{z(1+z'^2)}} = c'$$

$$\sqrt{z(1+z'^2)} = \sqrt{c}$$

$$z' = \sqrt{\frac{c}{z} - 1}$$

$$\frac{dz}{\sqrt{\frac{c}{z}-1}} = dx$$

$$\sqrt{\frac{z}{c-z}}\, dz = dx$$

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{z(x'^2+1)}} = \sqrt{k}$$

$$x'^2 = kz(x'^2+1)$$

$$(1-kz)x'^2 = kz$$

$$\frac{dx}{dz} = \sqrt{\frac{kz}{1-kz}}$$

$$\int_0^x dx = \int_0^z \sqrt{\frac{kz}{1-kz}}\, dz$$

same

$$1/c = \frac{1}{k}$$

$$x = \int_0^z \sqrt{\frac{kz}{1-kz}}\, dz$$

Try $kz = \sin^2\alpha$ $\Rightarrow$ $z=0 \Rightarrow \alpha = 0$

$k\,dz = 2\sin\alpha\cos\alpha\, d\alpha$

$$x = \int_0 \sqrt{\frac{\sin^2\alpha}{1-\sin^2\alpha}}\ \frac{2}{k}\sin\alpha\cos\alpha\, d\alpha$$

$$= \frac{2}{k}\int \sin^2\alpha\, d\alpha$$

$$kx = \int (1-\cos 2\alpha)\, d\alpha$$

$$= \alpha - \frac{\sin 2\alpha}{2}\Big|_0^\alpha$$

$$2kx = 2\alpha - \sin(2\alpha)$$

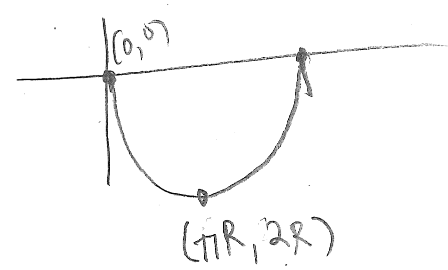
Let $2k = \frac{1}{R}$ and $2\alpha = \theta$

$\Rightarrow$ $\boxed{x = R(\theta - \sin\theta)}$

Then $kz = \sin^2\alpha = \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta)$

$\boxed{z = R(1-\cos\theta)}$

cycloid



<u>But what is R?</u>

$$x_f = R\,(\theta_f - \sin\theta_f)$$

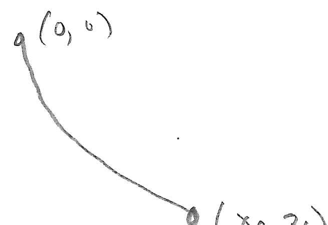
$$z_f = R\,(1 - \cos\theta_f)$$

$$\Rightarrow \quad \frac{z_f}{x_f} = \frac{1-\cos\theta_f}{\theta_f - \sin\theta_f}$$

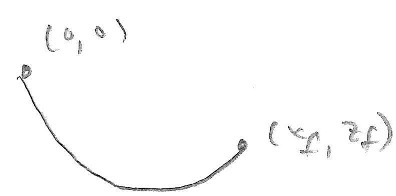
Given the slope between $(0,0)$ and (x_f, z_f) numerically solve this transcendental eqn for θ_f

Then solve $R = \dfrac{x_f}{\theta_f - \sin\theta_f}$ to find R.

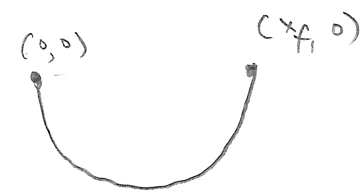
If $\dfrac{z_f}{x_f} > \dfrac{2}{\pi}$



If $\dfrac{z_f}{x_f} < \dfrac{2}{\pi}$



If $z_f = 0 \Rightarrow \theta_f = 2\pi \Rightarrow R = \dfrac{x_f}{2\pi}$



fastest path

Let's calculate the time

$$T = \int \sqrt{\frac{dx^2 + dz^2}{2gz}}$$

$$dx = R(1 - \cos\theta)\, d\theta$$

$$dz = R \sin\theta \, d\theta$$

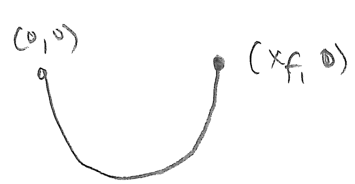
$$dx^2 + dz^2 = R^2\left[(1 - 2\cos\theta + \cos^2\theta) + \sin^2\theta\right] d\theta^2$$

$$= R^2 (2 - 2\cos\theta)\, d\theta^2$$

$$2gz = 2gR(1 - \cos\theta)$$

$$\frac{dx^2 + dz^2}{2gz} = \frac{R}{g}\, d\theta^2$$

$$T = \sqrt{\frac{R}{g}} \int d\theta = \sqrt{\frac{R}{g}}\, \theta_f$$



$$R = \frac{x_f}{2\pi}$$

$$\theta_f = 2\pi$$

$$T = \sqrt{2\pi \frac{x_f}{g}} \approx 2.517 \sqrt{\frac{x_f}{g}}$$

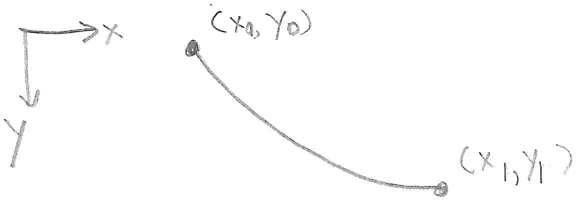
$x_f = 1.5\,\text{m} \Rightarrow T \approx 1$ sec

[For a circle $T = \underbrace{2.62}_{\sqrt{2}K(\frac{1}{2})} \sqrt{\frac{x_f}{g}}$, see pendulum]

June 2006 at Strings conf.

- Johann Bernoulli posed this as a challenge 1696
- Newton solved this in 1 day in 1696
  Leibniz, L'Hospital & Bernoullis also found it.
- Huygens ~~also~~ noted that cycloid is a tautochrone.
  (1673)

Brachistochrone



$$\frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_0^2 - mgy_0$$

$$v = \sqrt{v_0^2 + 2g(y-y_0)}$$

Let $v_0 = 0$ and $y_0 = 0$ $\Rightarrow v = \sqrt{2gy}$

Time of trajectory $T[y(x)] = \int \frac{ds}{v} = \int \frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}}$

$$\Rightarrow L = \sqrt{\frac{1+y'^2}{y}}, \quad \frac{\partial L}{\partial y} = -\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}}, \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{y(1+y'^2)}}$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{y''}{\sqrt{y(1+y'^2)}} - \frac{1}{2}\frac{y'^2}{\sqrt{y^3(1+y'^2)}} - \frac{y'^2 y''}{\sqrt{y(1+y'^2)^3}}$$

$$= \frac{y(1+y'^2)y'' - \frac{1}{2}y'^2(1+y'^2) - yy'^2y''}{\sqrt{y^3(1+y'^2)^3}}$$

$$= \frac{yy'' - \frac{1}{2}y'^2(1+y'^2)}{\sqrt{y^3(1+y'^2)^3}} = -\frac{1}{2}\frac{(1+y'^2)^2}{\sqrt{y^3(1+y'^2)}}$$

$$2yy'' - y'^2(1+y'^2) = -(1+y'^2)^2$$

$$\Rightarrow \boxed{2yy'' + 1 + y'^2 = 0}$$

Better to parametrize curve: $x(p), y(p)$

Then $y' = \frac{\dot{y}}{\dot{x}}, \quad y'' = \frac{\ddot{y}}{\dot{x}^2} - \frac{\dot{y}\ddot{x}}{\dot{x}^3} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$

$$\Rightarrow \boxed{2y(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \dot{x}^3 + \dot{x}\dot{y}^2 = 0}$$

Since $\frac{\partial L}{\partial x} = 0$ one can use Beltrami identity

$$L - y'\frac{\partial L}{\partial y'} = C$$

$$\sqrt{\frac{1+y'^2}{y}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} = C$$

$$\frac{1}{\sqrt{y(1+y'^2)}} = C$$

$$y(1+y'^2) = k^2$$

$$\Rightarrow \begin{cases} x = \frac{k^2}{2}(p - \sin p) \\ y = \frac{k^2}{2}(1 - \cos p) \end{cases}$$

Brachistochrone (cont)

$$T = \int \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{2gy}}\,dp$$

$$L = \sqrt{\frac{\dot{x}^2+\dot{y}^2}{y}}, \quad \frac{\partial L}{\partial y} = -\frac{1}{2}\sqrt{\frac{\dot{x}^2+\dot{y}^2}{y^3}}, \quad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}$$

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}$$

$$\frac{d}{dp}\left(\frac{\dot{x}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}\right) = 0 \quad \Rightarrow \quad \boxed{\dot{x}^2 = \frac{1}{2}K\,y(\dot{x}^2+\dot{y}^2)}$$

$$\hookrightarrow \frac{\ddot{x}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}} + \dot{x}\frac{d}{dp}\left(\frac{1}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}\right) = 0$$

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\ddot{y}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}} + \dot{y}\underbrace{\frac{d}{dp}\left(\frac{1}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}\right)}_{-\frac{\ddot{x}}{\dot{x}\sqrt{y(\dot{x}^2+\dot{y}^2)}}} = -\frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{\sqrt{y^3(\dot{x}^2+\dot{y}^2)}}$$

$$\ddot{y} - \frac{\dot{y}}{\dot{x}}\ddot{x} = -\frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y}$$

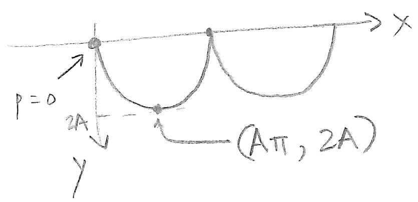
$$\boxed{2y(\dot{x}\ddot{y}-\dot{y}\ddot{x}) + \dot{x}(\dot{x}^2+\dot{y}^2) = 0}$$ as before

Brachistochrone (cont)

To show: cycloid = brachistochrone

$$\left.\begin{aligned} \dot{x}^2 &= \tfrac{1}{2}Ky(\dot{x}^2+\dot{y}^2) \\ 2y(\dot{x}\ddot{y}-\dot{y}\ddot{x}) &+ \dot{x}(\dot{x}^2+\dot{y}^2) = 0 \end{aligned}\right\} \quad Ky^2(\dot{x}\ddot{y}-\dot{y}\ddot{x}) + \dot{x}^3 = 0$$

Let $\begin{cases} x = A(p-\sin p) \\ y = A(1-\cos p) \end{cases} \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = A\sin p \end{cases} \Rightarrow \dot{x}^2+\dot{y}^2 = 2A^2(1-\cos p) = 2Ay$



Check $\dot{x}^2 = \frac{1}{2}Ky(\dot{x}^2+\dot{y}^2)$

$y^2 = \frac{1}{2}Ky(2Ay) \Rightarrow K = \frac{1}{A}$

Check $Ky^2(\dot{x}\ddot{y}-\dot{y}\ddot{x}) + \dot{x}^3 = 0$

$$\frac{1}{A}y^2\Big(\underbrace{A[1-\cos p][A\cos p] - [A\sin p]^2}_{\underbrace{A^2\cos p - A^2}_{-Ay}}\Big) + y^3 = 0 \checkmark$$

Hence $\frac{y_1}{x_1} = \left(\frac{1-\cos p_1}{p_1-\sin p_1}\right)$ determines p_1 $\rightarrow$ so if $\frac{y_1}{x_1} \gtrless \frac{2}{A}$ then $p_1 \lessgtr \pi$

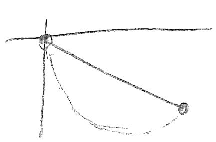
then $y_1 = A(1-\cos p_1)$ determines A ie

Then $T(x_1,y_1) = \int \sqrt{\frac{\dot{x}^2+\dot{y}^2}{2gy}}\,dp = \int_0^{p_1}\sqrt{\frac{A}{g}}\,dp = \sqrt{\frac{A}{g}}\,p_1$

along brachistochrone

(eg if $x_1 \to 0$, then $p_1 \Rightarrow 0$ so $y_1 \to A\frac{1}{2}p_1^2$ so $T(0,y_1) = \sqrt{\frac{2y_1}{g}}$ ✓)

Consider straight line path $\Rightarrow \begin{cases} x = x_1 p \\ y = y_1 p \end{cases} \Rightarrow T(x_1,y_1) = \sqrt{\frac{x_1^2+y_1^2}{2gy_1}}\int_0^1 \sqrt{p}\,dp = \sqrt{\frac{2(x_1^2+y_1^2)}{gy_1}}$



Also: $T = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2s}{g\sin\theta}} = \sqrt{\frac{2\sqrt{x_1^2+y_1^2}}{g\left(\frac{y_1}{\sqrt{x_1^2+y_1^2}}\right)}} = \sqrt{\frac{2(x_1^2+y_1^2)}{gy_1}}$ ✓

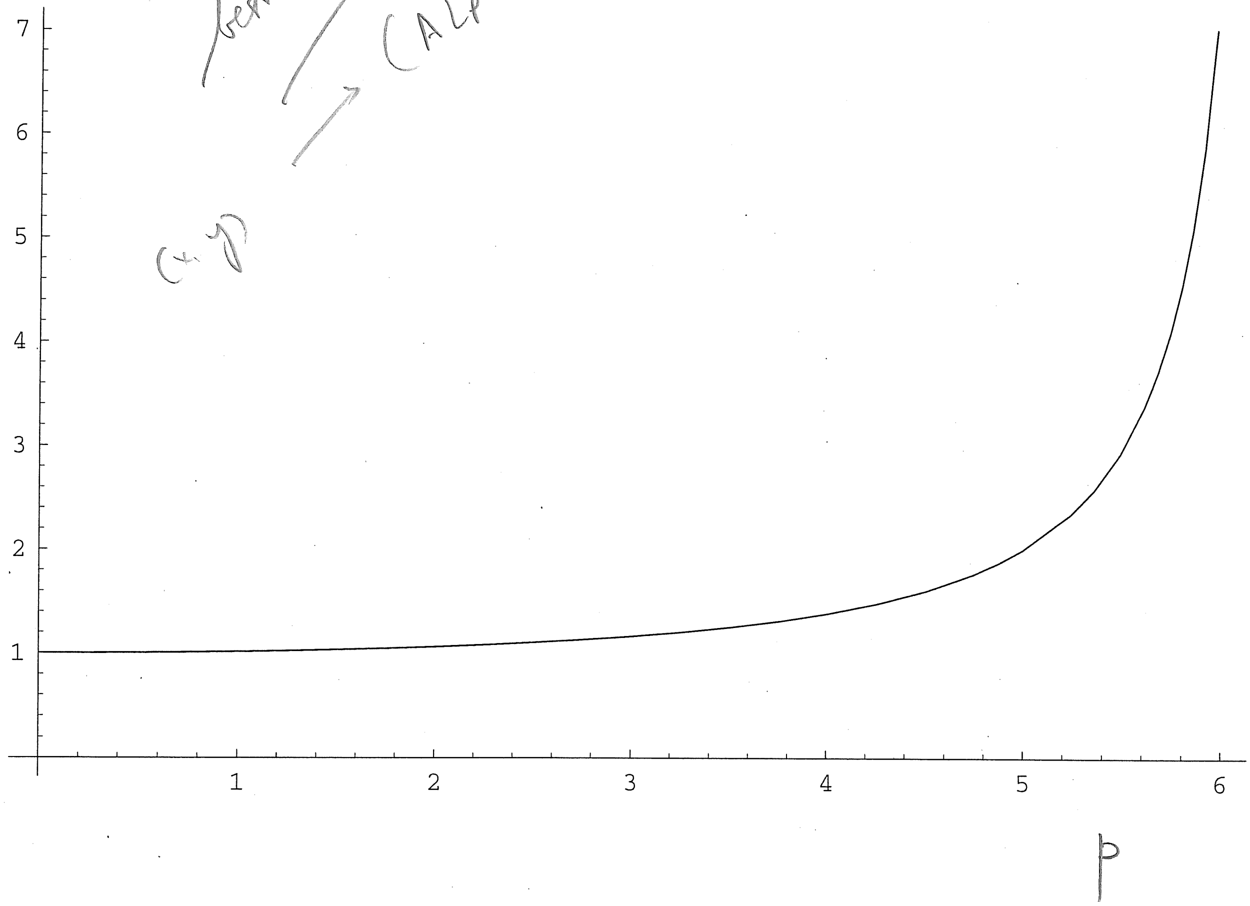
plot this.

Compare times: $\frac{T_{straight}}{T_{cyc}} = \frac{\sqrt{\frac{2(x_1^2+y_1^2)}{gy_1}}}{\sqrt{\frac{A}{g}p_1^2}} = \sqrt{\frac{2[(p-\sin p)^2+(1-\cos p)^2]}{p_1^2[1-\cos p]}} = \sqrt{\frac{2(p_1^2-2p_1\sin p_1+2(1-\cos p_1))}{p_1^2(1-\cos p_1)}}$

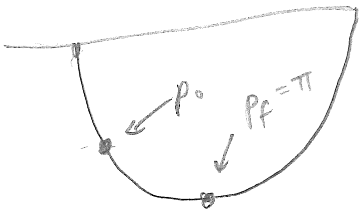
$(p_1 \to 0 \Rightarrow \text{ratio} \to 1)$

ratio of time between a straight line path + a brachistrone between (0,0) and $(A[p - \sin p], -A(1 - \cos p))$

(x, y)



Huygens noticed (1673) that a cycloid is ~~[illegible]~~ a <u>tautochrone</u>



$$X = A(p - \sin p)$$
$$Y = A(1 - \cos p)$$

Start at p_0, end at $p_f = \pi$

$$T = \int_{p_0}^{p_f} \sqrt{\frac{\dot{x}^2 + \dot{y}^2}{2g(y - y_0)}} = \sqrt{\frac{A}{g}} \int_{p_0}^{p_f} \sqrt{\frac{1 - \cos p}{\cos p_0 - \cos p}}\, dp$$

As calc'd before, $p_0 = 0 \Rightarrow T = \sqrt{\frac{A}{g}}\, \pi$

But now (help from Mathworld)

$$\cos p = 2\cos^2\left(\tfrac{p}{2}\right) - 1$$
$$1 - \cos p = 2\sin^2\left(\tfrac{p}{2}\right)$$

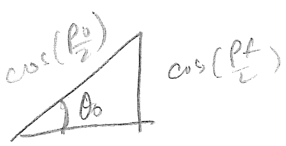
$$\Rightarrow \int_{p_0}^{p_f} \sqrt{\frac{1-\cos p}{\cos p_0 - \cos p}}\, dp = \int_{p_0}^{p_f} \frac{\sin\left(\frac{p}{2}\right) dp}{\sqrt{\cos^2\left(\frac{p_0}{2}\right) - \cos^2\left(\frac{p}{2}\right)}}$$

$$u = \frac{\cos\left(\frac{p}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}$$

$$= \int_{p_0}^{p_f} \frac{\frac{\sin\left(\frac{p}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}\, dp}{\sqrt{1 - \frac{\cos^2\left(\frac{p}{2}\right)}{\cos^2\left(\frac{p_0}{2}\right)}}} = \int_{1}^{\frac{\cos\left(\frac{p_f}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}} \frac{-2\, du}{\sqrt{1-u^2}}$$

$$du = -\frac{\frac{1}{2}\sin\left(\frac{p}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}$$

$$= 2 \arcsin u \Big|_{u = \frac{\cos\left(\frac{p_f}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}}^{\text{"}1\text{"}}$$

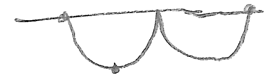


$$= \pi - \theta_0 \xrightarrow[\substack{p_f \to \pi \\ \cos\left(\frac{p_f}{2}\right) \to 0 \\ \theta_0 \to 0}]{} \pi, \text{ indep of } p_0$$

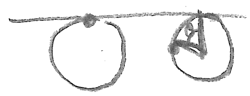
Thus time from p_0 to π is <u>indep of starting pt.</u>

~~[illegible]~~ $\boxed{\therefore \text{ amplitude-indep pendulum}}$

(4/18)



Cycloid



$$x = R(\theta - \sin\theta)$$
$$z = -R(1-\cos\theta)$$

Try to show a cycloid is a tautochrone using Lagrange approach

Let $\theta = \pi + \phi$

$$x = R(\pi + \phi - \sin(\pi+\phi))$$
$$= R\pi + R(\phi + \sin\phi)$$
$$z = -R(1 - \cos(\pi+\phi))$$
$$= -R - R\cos\phi$$
$$= -2R + R(1-\cos\phi)$$

Shift origin by $R\pi, -2R$

parametric path $\longrightarrow$
$$\begin{cases} x = R(\phi + \sin\phi) \approx 2R\phi \\ z = R(1-\cos\phi) \approx \frac{1}{2}R\phi^2 \end{cases}$$

$$\Rightarrow z \approx \frac{1}{8R}x^2 \Rightarrow U = mgz = \frac{mg}{8R}x^2$$

$$\omega^2 \approx \frac{g}{4R}$$

$$\dot{x} = R(1+\cos\phi)\dot{\phi}$$
$$\dot{z} = R\sin\phi\,\dot{\phi}$$

$$L = \frac{1}{2}mR^2\big[\underbrace{(1+\cos\phi)^2 + (\sin\phi)^2}_{2+2\cos\phi}\big]\dot{\phi}^2 - mgR(1-\cos\phi)$$

$$\cancel{\frac{\partial L}{\partial\dot\phi} = 2mR^2(1+\cos\phi)\dot\phi \quad \frac{\partial L}{\partial\phi} = -2mR^2\sin\phi\,\dot\phi^2 - mgR\sin\phi}$$

$$\dot\phi\frac{\partial L}{\partial\dot\phi} - L = mR^2(1+\cos\phi)\dot\phi^2 + mgR(1-\cos\phi) = E$$

$$dt = \int\sqrt{\frac{(1+\cos\phi)}{\epsilon - \frac{g}{R}(1-\cos\phi)}}\,d\phi = \int_0^{\phi_0} d\phi\,\frac{\sqrt{1+\cos\phi}}{\sqrt{\frac{g}{R}(\cos\phi - \cos\phi_0)}}$$